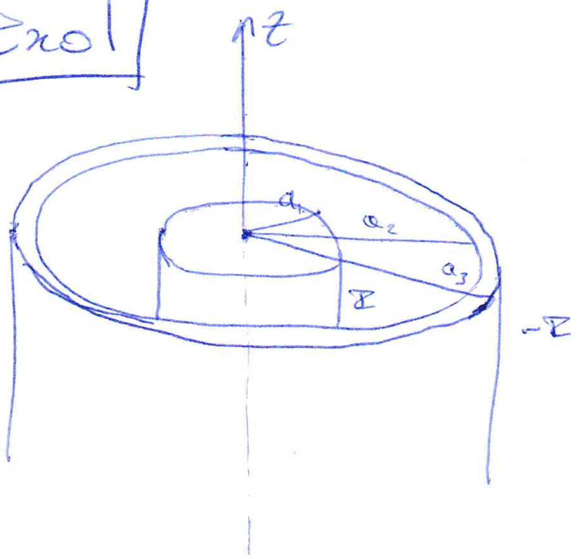
Séance 5 : Corrigés

Exo 1



Thm d'Ampère :

$$\oint_{\partial S} \vec{B}\cdot\vec{dl} = \mu_0 \iint_S \vec{J}\cdot\vec{dS}$$

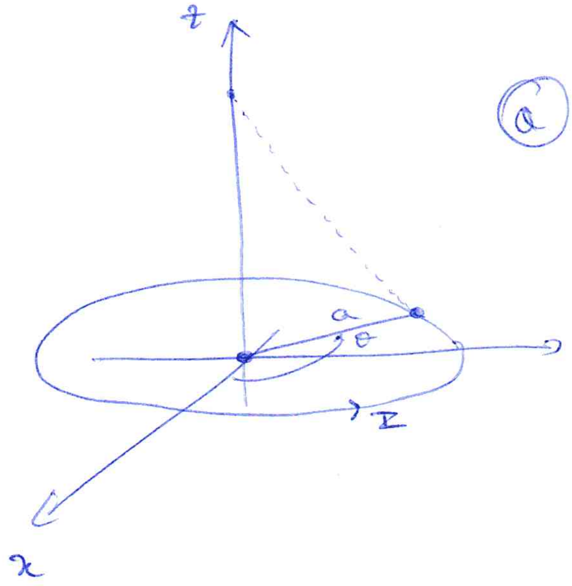
1/ Symétrie cylindrique : $\vec{B} = B(r)\,\vec{u_\theta}$ et $\vec{dl} = dl\,\vec{u_\theta}$
$= r\,d\theta\,\vec{u_\theta}$

$$\Rightarrow B(r) = \frac{\mu_0}{2\pi r} \underbrace{\iint_S \vec{J}\cdot\vec{dS}}_{I_{int}}$$

$$I_{int} = \begin{cases} I\cdot\left(\frac{r}{a_1}\right)^2 & \text{si } r \le a_1 \\ I & \text{si } a_1 \le r \le a_2 \\ I - I\,\frac{r^2 - a_2^2}{a_3^2 - a_2^2} & a_2 \le r \le a_3 \\ 0 & a_3 \le r \end{cases}$$

$$\Rightarrow \boxed{B(r) = \begin{cases} \frac{\mu_0 I}{2\pi r}\left(r/a_1\right)^2 & r \le a_1 \\ \frac{\mu_0 I}{2\pi r} & a_1 \le r \le a_2 \\ \frac{\mu_0 I}{2\pi r}\left(1 - \frac{r^2 - a_2^2}{a_3^2 - a_2^2}\right) & a_2 \le r \le a_3 \\ 0 & a_3 \le r \end{cases}}$$

Exo 2



(a) Biot & Savart:

$$\vec{dB} = \frac{\mu_0}{4\pi} \frac{I \, \vec{dl} \times \vec{r}}{|\vec{r}|^3}$$

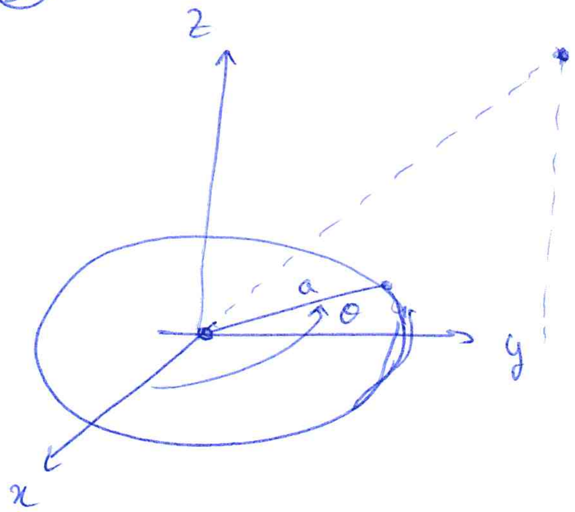
$$\begin{cases} \vec{dl} = a \, d\theta \, \vec{u}_\theta \\ \vec{r} = -a \, \vec{u}_r + z \, \vec{u}_z \end{cases}$$

$$\rightarrow \vec{dB} = \frac{\mu_0}{4\pi} \frac{I a \, d\theta}{(a^2+z^2)^{3/2}} \left[z \, \vec{u}_r + a \, \vec{u}_z \right]$$

$$\hookrightarrow \vec{B}(z) = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \left[\frac{z}{(a^2+z^2)^{3/2}} \cdot \underbrace{\vec{u}_r}_{2\pi\text{-périodique}} \xrightarrow{} 0 + \frac{a}{(a^2+z^2)^{3/2}} \cdot \vec{u}_z \right] d\theta$$

$$\Rightarrow \boxed{\vec{B}(z) = \frac{\mu_0 I}{2} \frac{a^2}{(a^2+z^2)^{3/2}} \vec{u}_z}$$

b



$$\begin{cases} \vec{dl} = a\,d\theta\,\vec{u_\theta} \\ \vec{r} = -a\vec{u_r} + z\vec{u_z} + y\vec{u_y} \\ \vec{u_y} = \cos\theta\,\vec{u_\theta} + \sin\theta\,\vec{u_r} \end{cases}$$

$$\rightarrow \vec{r} = (y\sin\theta - a)\vec{u_r} + z\vec{u_z} + y\cos\theta\,\vec{u_\theta}$$

$$\Rightarrow \vec{dB} = \frac{\mu_0}{4\pi}\frac{I\,a\,d\theta}{(a^2+z^2+y^2-2ay\sin\theta)^{3/2}}\left[z\vec{u_r} + (a - y\sin\theta)\vec{u_z}\right]$$

$$\vec{u_r} = \cos\theta\,\vec{u_x} + \sin\theta\,\vec{u_y}$$

$$\rightarrow \vec{B} = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{z\cos\theta\,\vec{u_x} + z\sin\theta\,\vec{u_y} + (a - y\sin\theta)\vec{u_z}}{(a^2+z^2+y^2-2ay\sin\theta)^{3/2}}\,d\theta$$

Soit $R^2 \equiv y^2 + z^2$

$$\Rightarrow \vec{B} = \frac{\mu_0 I a}{4\pi R^2}\int_0^{2\pi}\frac{\frac{z}{R}\cos\theta\,\vec{u_x} + \frac{z}{R}\sin\theta\,\vec{u_y} + \left(\frac{a}{R} - \frac{y}{R}\sin\theta\right)\vec{u_z}}{\left(\frac{a^2}{R^2} + 1 - 2\frac{ay}{R^2}\sin\theta\right)^{3/2}}\,d\theta$$

Soit $\vec{f}(\varepsilon) \equiv \frac{1}{\left(\varepsilon^2 + 1 - 2\varepsilon\frac{y\sin\theta}{R}\right)^{3/2}}\left(\frac{z}{R}\cos\theta,\ \frac{z}{R}\sin\theta,\ \varepsilon + \frac{y\sin\theta}{R}\right)$

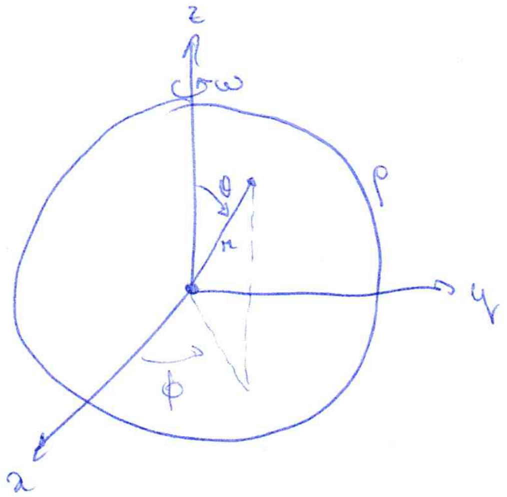
$$\vec{f}(\varepsilon) \underset{\varepsilon \ll 1}{\simeq} \frac{1}{R}\Big(\underset{\substack{\uparrow \\ 2\pi - \rho}}{z\cos\theta} + 3\frac{yz}{R}\cos\theta\sin\theta.\,\varepsilon,\ \underset{\substack{\uparrow \\ 2\pi - \rho}}{z\sin\theta} + 3\frac{yz}{R}\sin^2\theta\,\varepsilon,\ \underset{\substack{\uparrow \\ 2\pi - \rho}}{y\sin\theta} + \left(R + \frac{3y^2\sin^2\theta}{R^2}\right)\varepsilon\Big)$$

$$\vec{B} = \frac{\mu_0 I a}{4\pi R^2}\int_0^{2\pi} \vec{f}(\varepsilon)\, d\theta$$

$$\simeq \frac{\mu_0 I a}{4\cancel{\pi} R^2}\left(0,\ \frac{3\cancel{\pi} y z \varepsilon}{R^2},\ \cancel{\pi}\left(2+\frac{3y^2}{R^2}\right)\varepsilon\right)$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0 I a^2}{4R^3}\left[3\frac{y}{R}\frac{z}{R}\vec{u}_y + 2\left(1+\frac{3y^2}{2R^2}\right)\vec{u}_z\right]} \quad \text{Si } a \ll R$$

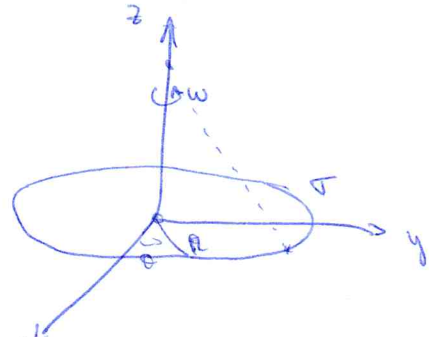
Exo 3



$$\vec{J} = \rho\vec{v} = \rho\,\vec{\omega}\times\vec{r}$$
$$= \rho\omega r \sin\theta\, \vec{u}_\phi$$

$$\Rightarrow \boxed{\vec{J} = \rho\omega r \sin\theta\, \vec{u}_\phi}$$

Exo 4



$$dI = \vec{j}\cdot\vec{dl}$$
$$= \sigma\vec{v}\cdot\vec{dl}$$
$$= \sigma(\vec{\omega}\times\vec{r})\cdot dr\,\vec{u}_\theta$$
$$= \sigma\omega r\, dr$$

$$\Rightarrow d\vec{B} = \frac{\mu_0}{4\pi}\,\sigma\omega r\, dr \cdot \frac{\vec{dl}\times\vec{r}}{|\vec{r}|^3}$$

$$\begin{cases} \vec{dl} = -r\,d\theta\,\vec{u_\theta} \\ \vec{r} = -r\,\vec{u_r} + z\,\vec{u_z} \end{cases}$$

$$\hookrightarrow \vec{B} = \frac{\mu_0 \sigma \omega}{4\pi} \int \frac{r}{|\vec{r}|^3} \left(r^2 \int_0^{2\pi} d\theta\, \vec{u_z} - r z \underbrace{\int_0^{2\pi} d\theta\, \vec{u_r}}_{0} \right) dr$$

$$= \frac{\mu_0 \sigma \omega}{4\pi}\, 2\pi \int \frac{r^3}{(r^2+z^2)^{3/2}}\, dr\, \vec{u_z}$$

$$\Rightarrow \boxed{\vec{B}(z) = \begin{cases} \frac{1}{2}\mu_0 \sigma \omega \dfrac{\left(z - \sqrt{a^2+z^2}\right)^2}{\sqrt{a^2+z^2}}\,\vec{u_z} & \text{si } z \geq 0 \\ \frac{1}{2}\mu_0 \sigma \omega \dfrac{\left(z + \sqrt{a^2+z^2}\right)^2}{\sqrt{a^2+z^2}}\,\vec{u_z} & \text{si } z < 0 \end{cases}}$$

Exo 5

$$\begin{cases} \vec{B} = (0, 0, B_z) \\ \vec{A} = (A_x, A_y, A_z) \end{cases}$$

$$\vec{\nabla} \times \vec{A} = \vec{B} \Rightarrow \begin{cases} 0 = \partial_y A_z - \partial_z A_y \\ 0 = \partial_z A_x - \partial_x A_z \\ B_z = \partial_x A_y - \partial_y A_x \end{cases}$$

Solu$^°$ possibles : $\vec{A_1} = (0, x B_z, 0)$, $\vec{A_2} = (-y B_z, 0, 0)$, ...

$$\vec{A_1} = \vec{A_2} + \vec{\nabla}\psi \Rightarrow \vec{\nabla}\psi = (y B_z, x B_z, 0) = (y, x, 0) B_z$$

$$\Rightarrow \begin{cases} \partial_x \psi = y B_z \\ \partial_y \psi = x B_z \\ \partial_z \psi = 0 \end{cases} \longrightarrow \boxed{\psi(x, y, z) = x y B_z}$$

$\vec{A} = \frac{1}{2} \vec{B} \times \vec{x} =$

$\hookrightarrow A_k = \frac{1}{2} \epsilon_{ijk} B_i x_j$

$[\vec{\nabla} \times \vec{A}]_n = \epsilon_{lmn} \partial_l A_m$

$= \epsilon_{lmn} \partial_l \left(\frac{1}{2} \epsilon_{ijm} B_i x_j \right)$

$= \frac{1}{2} \epsilon_{lmn} \epsilon_{ijm} B_i \partial_l x_j$ $\quad \downarrow B_i = c^{ste}$

$= \frac{1}{2} \underbrace{\epsilon_{mnl} \epsilon_{mij}} B_i \partial_l x_j$

$= \frac{1}{2} \left(\delta_{ni} \delta_{lj} - \delta_{nj} \delta_{li} \right) B_i \partial_l x_j$

$= \frac{1}{2} \left[B_n \partial_j x_j - B_i \partial_i x_n \right]$

$= \frac{1}{2} \left[B_n (\vec{\nabla} \cdot \vec{x}) - \vec{B} \cdot \vec{\nabla}(x_n) \right]$

Mais : $\partial_i x_j = \delta_{ij} \Rightarrow \partial_i x_i = \delta_{ii} = 3$

$\Rightarrow [\vec{\nabla} \times \vec{A}]_n = \frac{1}{2} \left(B_n \cdot 3 - B_i \partial_i x_n \right)$

$= \frac{1}{2} \left(3 B_n - B_i \delta_{in} \right)$

$= \frac{1}{2} \left(3 B_n - B_n \right) = B_n$

$\Rightarrow \boxed{\vec{\nabla} \times \vec{A} = \vec{B}}$

ok.

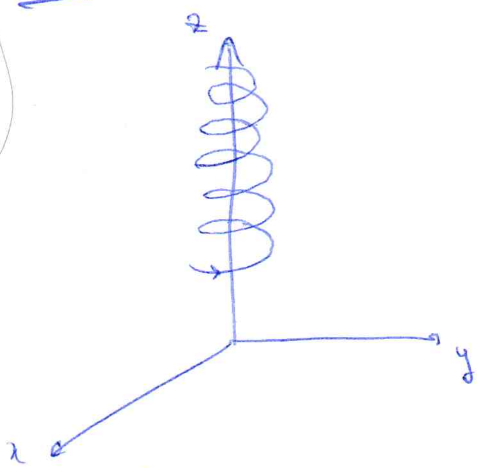
Exo 6

$\vec{B} = \vec{\nabla} \times \vec{A}$ car statique

$$\Rightarrow \int_S \vec{dS} \cdot \vec{B} = \int_S \vec{dS} \cdot (\vec{\nabla} \times \vec{A}) = \oint_{\partial S} \vec{dl} \cdot \vec{A} \quad \text{P/ Thm de Stockes.}$$

$$\hookrightarrow \int_S \vec{dS} \cdot \vec{B} = \oint_{\partial S} \vec{dl} \cdot \vec{A}$$

Exo 7



$$\begin{cases} \vec{B}_{in} = (0, 0, B_z) \\ \vec{B}_{out} = \vec{0} \\ \vec{A} = \dfrac{C}{x^2+y^2}(-y, x, 0) \end{cases}$$

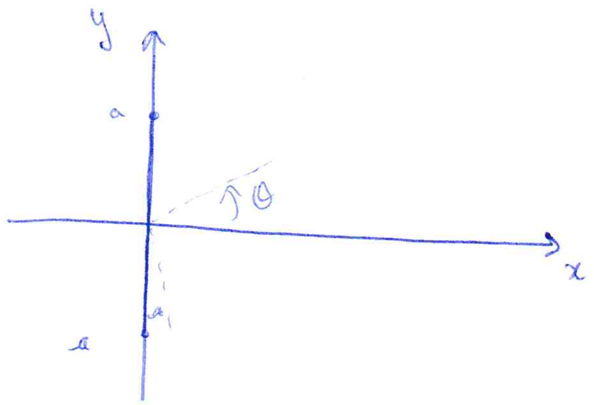
$$\vec{\nabla} \times \vec{A} = C \begin{pmatrix} 0 \\ 0 \\ \partial_x A_y - \partial_y A_x \end{pmatrix} = \vec{0}. \quad \text{ok.}$$

$$\begin{cases} \vec{u_x} = \cos\theta \, \vec{u_r} - \sin\theta \, \vec{u_\theta} \\ \vec{u_y} = \sin\theta \, \vec{u_r} + \cos\theta \, \vec{u_\theta} \\ \vec{u_z} = \vec{u_z} \end{cases} \qquad \vec{dl} = r \, d\theta \, \vec{u_\theta} \quad \text{et} \quad \vec{A} = \frac{C}{r} \vec{u_\theta}$$

$$\Rightarrow \oint_{\partial S} \vec{A} \cdot \vec{dl} = \int_0^{2\pi} r \, d\theta \, \vec{u_\theta} \cdot \frac{C}{r} \vec{u_\theta} = 2\pi C = \int_S \vec{dS} \cdot \vec{B} = B_z \cdot \pi a^2 = \phi$$

$$\Rightarrow \boxed{C = \frac{\phi}{2\pi} = \frac{B_z a^2}{2}}$$

$\boxed{\text{Ex } n^o 8}$



$\uparrow \vec{B}_{ext}$

$$\begin{cases} \vec{B} = B\vec{u_y} = B(\sin\theta\,\vec{u_r} + \cos\theta\,\vec{u_\theta}) \\ \vec{dS} = dS\,\vec{u_\theta} = dz\,dr\,\vec{u_\theta} \end{cases}$$

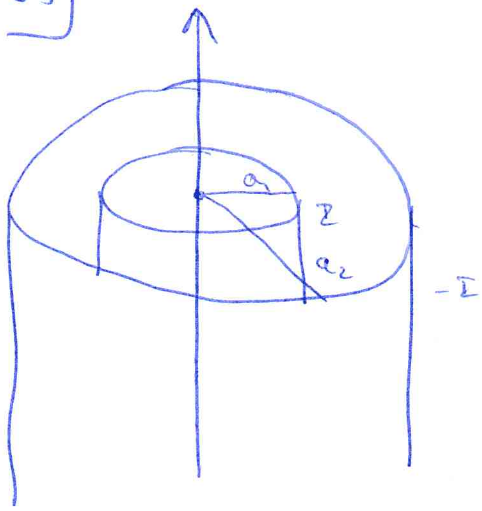
$$\Rightarrow \quad \phi_m = \int_S \vec{B}\cdot\vec{dS} = \int B\cos\theta \int_{-\frac{a}{2}}^{a/2} dz \int_{-a/2}^{a/2} dr = Ba^2\cos\theta$$

$$\vec{m} = \frac{1}{2} I \oint \vec{r}\times\vec{dl} = \frac{1}{2} I \left[2a\int_{-a/2}^{a/2} dz + 2a\int_{-a/2}^{a/2} dy \right] \vec{u_x}$$

$$= 2a^2 I\,\vec{u_x}$$

$$\vec{T} = \vec{m}\times\vec{B} = 2a^2 I B\,\vec{u_z}$$

Exo 9



$$I_{int} = \begin{cases} I & \text{si } a_1 \leq r < a_2 \\ 0 & \text{sinon} \end{cases}$$

$$\Rightarrow \oint_{\partial S} \vec{B} \cdot \vec{dl} = \mu_0 \iint_S \vec{J} \cdot \vec{dS}$$

$$\Rightarrow \vec{B} = \begin{cases} \frac{\mu_0 I}{2\pi r} \vec{u_\theta} & \text{si } a_1 \leq r < a_2 \\ \vec{0} & \text{sinon} \end{cases}$$

$$W_B = \frac{1}{2\mu_0} \iiint_V |\vec{B}|^2 dV \; , \quad w_B \equiv \frac{W_B}{L}$$ (énergie p/unité de longueur).

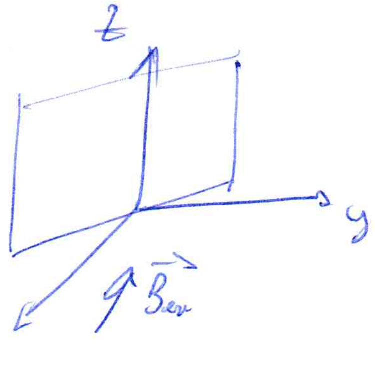
$$w_B = \frac{1}{2\mu_0} \int_0^{2\pi} \int_0^{\infty} |\vec{B}|^2 r \, dr \, d\theta$$

$$= \frac{1}{2\cancel{\mu_0}} \int_0^{2\pi} \int_{a_1}^{a_2} \frac{\mu_0^{\cancel{2}} I^2}{(2\pi)^2 r^2} r \, dr \, d\theta$$

$$= \frac{1}{2} \cdot 2\pi \cdot \frac{1}{(2\pi)^2} \cdot \mu_0 I^2 \int_{a_1}^{a_2} \frac{dr}{r}$$

$$\Rightarrow \boxed{w_B = \frac{\mu_0 I^2}{4\pi} \ln\left(a_2/a_1\right)}$$

$\boxed{\text{Ejercicio 10}}$



$$\mathcal{E} = -\frac{d\phi_m}{dt} = -\frac{d}{dt}\left(\iint \vec{B}\cdot d\vec{S}\right)$$

$$\begin{cases} \vec{B} = -B\vec{u}_x = -B(\vec{u}_r \cos\theta - \vec{u}_\theta \sin\theta) \\ d\vec{S} = r\,dr\,d\theta\,\vec{u}_r \end{cases}$$

$$\Rightarrow \mathcal{E} = \frac{d}{dt}\left\{B\iint r\cos\theta\,dr\,d\theta\right\}$$

$$= \frac{d}{dt}\left\{B\int \cos\theta\,d\theta \cdot \int_0^a r\,dr\right\}$$

$$= \frac{d}{dt}\left\{B\sin\theta \cdot \frac{1}{2}a^2\right\}\Bigg|_{\theta=\omega t}$$

$$= \frac{1}{2}Ba^2\omega\cos\omega t$$

$$\Rightarrow \boxed{\mathcal{E}(t) = \frac{1}{2}Ba^2\omega\cos(\omega t)}$$