5 Magnetostatics

1. Calculate everywhere in space the magnetic field produced by a coaxial cable composed of a conducting cylinder or radius a_1, and a hollow cylinder of internal radius $a_2 > a_1$ and external radius a_3. The internal and external cylinders carry total currents I and $-I$, respectively. Both currents are uniformly distributed over the section of the conductors.
2. Calculate at the point $(0, 0, z)$ the magnetic field of a circular loop of radius a centered around the origin in the (x, y) plane. Consider the limit $z \gg a$. Calculate the leading contribution to the magnetic field at $|\vec{x}| \gg a$ everywhere in space. Compare to the electric field of a dipole.
3. A uniformly charged sphere with charge density ρ centered at the origin is rotating around the z-axis with constant angular velocity ω. Find the resulting current density $\vec{J}(x)$.
4. A uniformly charged disk of radius a centered at the origin in the plane $z = 0$ is rotating around z-axis with constant angular velocity ω. Find the magnetic field at points of the z-axis. Find the leading contribution to the field at $|\vec{x}| \gg a$.
5. Consider the magnetic field $\vec{B} = (0, 0, B_z)$ or constant B_z. Give several vector potentials $\vec{A}$ which give rise to this magnetic field. Check that they differ by a gauge transformation $\vec{A} \to \vec{A} + \vec{\nabla}\Psi$.
   Consider now a general constant magnetic field $\vec{B}$. Can the vector potential be given by
   $$\vec{A} = \frac{1}{2}\vec{B} \times \vec{x}\,?$$
6. Let S be a piece of a 2-dimensional surface with boundary ∂S situated in a region of space filled with a static non-uniform magnetic field $\vec{B}(\vec{x})$. Demonstrate the relation
   $$\int_S d\vec{s} \cdot \vec{B} = \int_{\partial S} d\vec{l} \cdot \vec{A}.$$
   In other words, the flux of the magnetic field through any surface is equal to the line integral of the vector potential along the boundary of this surface.
7. Consider a solenoïde of radius a with its axis directed along the z-axis. Let the constant magnetic field inside the solenoïde be $\vec{B} = (0, 0, B_z)$, while the magnetic field outside be zero, $\vec{B}_{\text{out}} = 0$. Does the vector potential vanish outside of the solenoïde ? Can it be represented as the gradient of a scalar function ? Can the following expression represent the potential outside of the solenoïde,
   $$\vec{A} = \frac{C}{x^2 + y^2}(-y, x, 0).$$
   Calculate the integral of $\vec{A}$ along the circular contour going around the solenoïde outside of it and relate the constant C to the flux of the magnetic field.
8. A square conducting frame with sides of length a carries a current I. The frame is placed in a uniform magnetic field $\vec{B}$ in such a way that one pair of opposite sides is perpendicular to the magnetic field. Calculate the moment of force (torque, *couple*) acting on the frame for different orientations of the frame with respect to the magnetic field. Express the result in terms of the magnetic moment of the frame.

9. Two coaxial hollow cylinders of radii a_1 and $a_2 > a_1$ carry total currents I and $-I$, respectively. The currents are uniformly distributed over the surfaces of the cylinders. Calculate the energy, per unit length of the cylinders, contained in the magnetic field produced by these currents.

10. A square conducting frame with the side a is placed in a uniform magnetic field $\vec{B}$ perpendicular to the field. The frame starts to rotate with an angular velocity $\vec{\omega}$ perpendicular to $\vec{B}$. Find the electromotive force *emf* induced in the frame as a function of time. Does it depend on the position and orientation of the rotation axis ?