Séance 4 : Corrigés

Exo 1

(a) $\vec{\nabla}(\vec{a}\cdot\vec{b}) = \partial_i \vec{e_i}\,(\delta_{jk} a_j b_k)$

$= \partial_i (a_k b_k)\,\vec{e_i}$

$= (\partial_i a_k)\, b_k \vec{e_i} + a_k (\partial_i b_k)\,\vec{e_i}$

$(\vec{a}\cdot\vec{\nabla})\vec{b} + (\vec{b}\cdot\vec{\nabla})\vec{a} + \vec{a}\times(\vec{\nabla}\times\vec{b}) + \vec{b}\times(\vec{\nabla}\times\vec{a})$

$= a_i \partial_i b_j \vec{e_j} + b_i \partial_i a_j \vec{e_j} + \epsilon_{ijk} a_i (\epsilon_{mnj} \partial_m b_n \vec{e_k}) + \epsilon_{ijk} b_i (\epsilon_{mnj} \partial_m a_n \vec{e_k})$

$= a_i \partial_i b_j \vec{e_j} + b_i \partial_i a_j \vec{e_j} + a_i \partial_m b_n \vec{e_k} (\delta_{km}\delta_{in} - \delta_{kn}\delta_{im}) + b_i \partial_m a_n \vec{e_k} (\delta_{km}\delta_{in} - \delta_{kn}\delta_{im})$

$= a_i (\partial_k b_i)\vec{e_k} + b_i (\partial_k a_i)\vec{e_k}$

$\square$.

(b)

(c) Idem.

Exo 2

(a) $\vec{\nabla}\times(\vec{\nabla}\phi) = \epsilon_{ijk}\partial_i(\vec{\nabla}\phi)_j \vec{e_k} = \overbrace{\epsilon_{ijk}}^{\text{Antisymétrique}}\overbrace{\partial_i\partial_j}^{\text{Symétrique}}\phi\, \vec{e_k} = \vec{0}$

(b) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = \partial_i\, \epsilon_{jki}\partial_j A_k = \underbrace{\epsilon_{jki}}_{A-\text{sym}}\underbrace{\partial_i\partial_j}_{\text{sym}} A_k = 0$

Prop :

Soient A_{ij} antisymétrique ($A_{ij} = -A_{ji}$)

et S_{ij} symétrique ($S_{ij} = S_{ji}$)

$\Rightarrow \quad A_{ij} S_{ij} = 0$

Dém :

$$A_{ij} S_{ij} \overset{S \text{ sym}}{=} A_{ij} S_{ji} \overset{A \text{ A-sym}}{=} -A_{ji} S_{ji} \overset{\substack{j \to k \\ i \to l}}{=} -A_{kl} S_{kl}$$

$$\overset{\substack{k \to i \\ l \to j}}{=} -A_{ij} S_{ij}$$

$$\Rightarrow A_{ij} S_{ij} = -A_{ij} S_{ij} \Rightarrow 2 A_{ij} S_{ij} = 0$$

$$\Rightarrow \boxed{A_{ij} S_{ij} = 0} \quad \square.$$

$\boxed{\text{Exo 3}}$

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$$

$$\begin{cases} x = r \cos\theta \sin\phi \\ y = r \sin\theta \sin\phi \\ z = r \cos\phi \end{cases} \Rightarrow \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \cos\theta \sin\phi & -r \sin\theta \sin\phi & r \cos\theta \cos\phi \\ \sin\theta \sin\phi & r \cos\theta \sin\phi & r \sin\theta \cos\phi \\ \cos\phi & 0 & -r \sin\phi \end{pmatrix}$$

$$\times \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}$$

②

$$\longrightarrow \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ -\frac{\cos\phi\sin\theta}{r} & \frac{\cos\theta\cos\phi}{r} & 0 \\ \frac{\cos\theta\cos\phi}{r} & \frac{\cos\phi\sin\theta}{r} & -\frac{\sin\phi}{r} \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

$$\Rightarrow \frac{\partial}{\partial x} = \partial_x r\,\partial_r + \partial_x\theta\,\partial_\theta + \partial_x\phi\,\partial_\phi$$

$$= \cos\theta\sin\phi\,\partial_r - \frac{\cos\phi\sin\theta}{r}\partial_\theta + \frac{\cos\theta\cos\phi}{r}\partial_\phi$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} = \dots$$

$$\hookrightarrow \frac{\partial^2}{\partial y^2} = \dots$$

$$\frac{\partial^2}{\partial z^2} = \dots$$

$$\hookrightarrow \boxed{\Delta = \partial_r^2 + \frac{1}{r^2\sin^2\phi}\partial_\theta^2 + \frac{1}{r^2}\partial_\phi^2.}$$

$\boxed{\text{Exo 4}}$

$$\left| \begin{array}{l} \vec{\nabla}\times\vec{A} = \vec{0} \longrightarrow \text{ok}. \\ \vec{\nabla}\times\vec{B} \neq \vec{0} \longrightarrow \cancel{\text{ok}} \end{array} \right.$$

$$\vec{E} = -\vec{\nabla}\phi_A \Rightarrow \vec{\nabla}\cdot\vec{E} = -\Delta\phi_A = \frac{\rho_A}{\epsilon_0}$$

$$\Rightarrow \rho_A(x,y,z) = \vec{\nabla}\cdot\vec{E}_A\cdot\epsilon_0 = 2A_0\epsilon_0 xy.$$

$$\vec{E} = -\vec{\nabla}\phi$$

$$\begin{cases} \partial_x \phi = -A_0 (yz + yz^2) \\ \partial_y \phi = -A_0 (xz + xz^2) \\ \partial_z \phi = -A_0 (xy + 2xyz) \end{cases}$$

$$\phi_A = -A_0(xyz + xyz^2) + K_1(y,z)$$

$$= -A_0(xyz + xyz^2) + K_2(x,z)$$

$$= -A_0(xyz + xyz^2) + K_3(x,y)$$

$$\Rightarrow k_1 = K_2 = K_3 = 0$$

$$\hookrightarrow \boxed{\phi_A(x,y,z) = -A_0(xyz + xyz^2).}$$

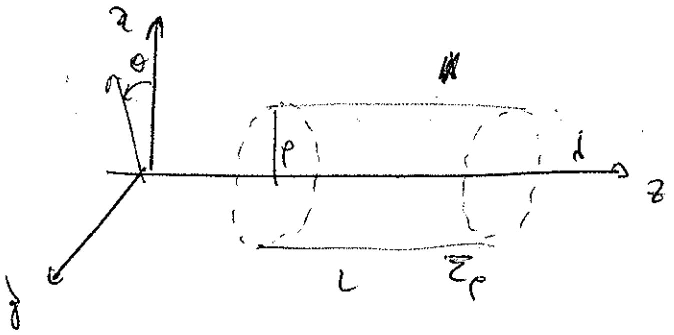
$\boxed{\text{Exo 5}}$

$$\vec{E} = (E(x,y,z), 0, 0)$$

ⓐ $\vec{\nabla} \times \vec{E} = \vec{0} \Rightarrow \begin{cases} \partial_z E = 0 \\ \partial_y E = 0 \end{cases} \Rightarrow \vec{E} = (E(x), 0, 0)$

ⓑ $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \underset{\rho = 0}{=} 0 \Leftrightarrow \partial_x E = 0 \Rightarrow \vec{E} = (E, 0, 0)$ □

Exo 6



a) $\vec{E} = E_r(r,\theta,z)\,\vec{e_r} + E_\theta(r,\theta,z)\,\vec{e_\theta} + E_z(r,\theta,z)\,\vec{e_z}$

$= E_r(r)\,\vec{e_r}$ p/ Invariance p/ rotat° et translat° selon Oz.

Gauss : $\oint\!\!\oint_{\Sigma_\rho} \vec{E}\cdot\vec{dS} = \oint\!\!\oint_{\Sigma_\rho} E(\rho)\cdot\vec{e_r}\cdot dS\cdot\vec{e_r} = E(\rho)\underbrace{\oint\!\!\oint_{\Sigma_\rho} dS}_{= S_\rho = 2\pi\rho\cdot L}$

$$= \frac{Q_{\Sigma_\rho}}{\epsilon_0} = \frac{\lambda\cdot L}{\epsilon_0}$$

$\Leftrightarrow\quad E(\rho) = \dfrac{\lambda}{2\pi\rho\epsilon_0} \longrightarrow \boxed{\vec{E}(\vec{r}) = \dfrac{\lambda}{2\pi r\epsilon_0}\,\vec{e_r}}$

b) $\vec{E} = -\vec{\nabla}\phi = -\left(\partial_r, \frac{1}{r}\partial_\theta, \partial_z\right)\phi$

$$\Rightarrow \begin{cases} \partial_\theta\phi = 0 \\ \partial_z\phi = 0 \\ \partial_r\phi = -\dfrac{\lambda}{2\pi r\epsilon_0} \end{cases} \longrightarrow \phi(r) = K + \frac{\lambda}{2\pi\epsilon_0}\ln(r)$$

$\boxed{\text{Exo 7}}$

Dans les 3 cas : $\vec{E}(r,\theta,\phi) = E(r)\,\vec{e_r}$ p/ symétrie sphérique.

$$\Rightarrow \oint_{\Sigma_\rho} \vec{E}\cdot\vec{dS} = E(\rho)\oint_{\Sigma_\rho} dS = 4\pi\rho^2 E(\rho) \underset{\text{Gauss}}{=} \frac{Q_{\Sigma_\rho}}{\epsilon_0}$$

$$\Rightarrow \vec{E} = \frac{Q_{\Sigma_\rho}}{4\pi\rho^2\epsilon_0}\vec{e_r}$$

ⓐ $\rho(r) = \begin{cases} \rho_0 & r < R \\ 0 & r \geq R \end{cases} \rightarrow Q_{\Sigma_\rho} = \begin{cases} \rho_0 \frac{4}{3}\pi\rho^3 & \rho < R \\ \rho_0 \frac{4}{3}\pi R^3 & \rho \geq R \end{cases}$

$$\rightarrow \vec{E} = \begin{cases} \frac{\rho_0}{3\epsilon_0} r\,\vec{e_r} & r < R \\ \frac{\rho_0}{3\epsilon_0}\frac{R^3}{r^2}\vec{e_r} & r \geq R \end{cases}$$

$$\phi(r) = \begin{cases} \frac{\rho_0}{6\epsilon_0}(3R^2 - r^2) & r < R \\ \frac{\rho_0}{3\epsilon_0}\frac{R^3}{r} & r \geq R \end{cases}$$

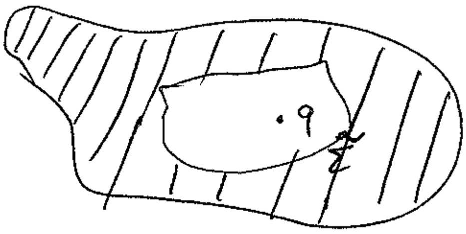
b) $\rho(r) = \begin{cases} C/r & r < R \\ 0 & r \geq R \end{cases}$

$$\rightarrow \left[\begin{array}{l} \vec{E} = \begin{cases} \dfrac{C}{3\varepsilon_0}\,\vec{e_r} & r < R \\[2ex] \dfrac{R^3 C}{r^3 3\varepsilon_0}\,\vec{e_r} & r \geq R \end{cases} \\[6ex] \phi(r) = \begin{cases} \dfrac{C}{6\varepsilon_0}(3R - 2r) & r < R \\[2ex] \dfrac{C}{6\varepsilon_0}\,\dfrac{R^3}{r^2} & r \geq R \end{cases} \end{array} \right.$$

c) $\sigma(r) = \begin{cases} 0 & r \neq R \\ \sigma & r = R \end{cases} \rightarrow \Phi_{\varepsilon\rho^2} \begin{cases} 0 & r < R \\ 4\pi R^2 \sigma & r \geq R \end{cases}$

$$\rightarrow \left[\begin{array}{l} \vec{E} = \begin{cases} \vec{0} & r < R \\[2ex] \dfrac{\sigma}{\varepsilon_0}\,\dfrac{R^2}{r^2}\,\vec{e_r} & r \geq R \end{cases} \\[6ex] \phi = \begin{cases} \dfrac{\sigma}{\varepsilon_0} R & r < R \\[2ex] \dfrac{\sigma}{\varepsilon_0}\,\dfrac{R^2}{r} & r \geq R \end{cases} \end{array} \right.$$

$\boxed{\text{Exo 8}}$



Gauss : $\displaystyle\oiint_{\Sigma} \vec{E}\cdot\vec{dS} = \frac{Q_\Sigma}{\epsilon_0}$

3 régions

$\Sigma \subset \tilde\Sigma$: $\vec{E} \neq \vec{0}$

$\left.\begin{array}{l}\Sigma = \tilde\Sigma \\ \Sigma \supset \tilde\Sigma\end{array}\right\}$ $\vec{E} = \vec{0}$ car dans un conducteur.

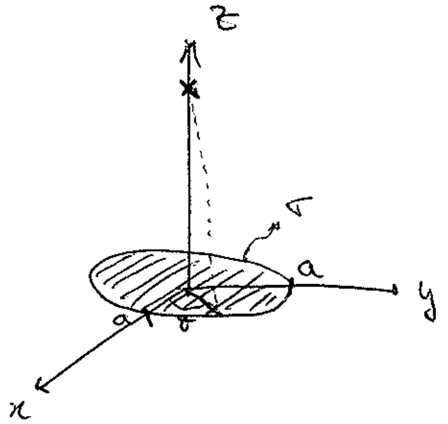
or $\displaystyle\oiint_{\Sigma} \vec{E}\cdot\vec{dS} = \frac{Q_\Sigma}{\epsilon_0} = \frac{q + q_{ind}}{\epsilon_0} = 0$

$\Leftrightarrow \boxed{q_{ind} = -q}$

q_{ind} / $\tilde\Sigma$ car on peut prendre Σ aussi proche de $\tilde\Sigma$ que l'on veut.

$\square$.

$\boxed{\text{Exo 9}}$



$$\vec{dE} = \frac{dQ}{4\pi\varepsilon_0}\,\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

$$dQ = \sigma(\vec{r}')\, r' dr' d\theta$$

$$\rightarrow \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_0^a\int_0^{2\pi}\sigma(\vec{r}')\, r'\,\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}\, dr' d\theta$$

- $\sigma(\vec{r}') = \sigma$ (Uniforme)
- $\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \equiv \frac{\vec{u}}{u^3}$ $\quad (\vec{u} \equiv \vec{r}-\vec{r}')$ $\quad \vec{u} = u\,\vec{e}_u$ $\left(\vec{e}_u = \vec{e}_z \frac{|z|}{u}\right)$

$\rightarrow$

$\hookrightarrow r'dr' = u\,du$

$\hookrightarrow dr' = \frac{u\,du}{\sqrt{u^2-z^2}}$

$$\rightarrow E = \frac{\sigma}{2\varepsilon_0}\int_z^{\sqrt{z^2+a^2}} \frac{z\,du}{u}$$

$$\boxed{\vec{E}(z) = \begin{cases} \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{1}{\sqrt{1+a^2/z^2}}\right)\vec{e}_z & \text{si } z > 0 \\ -\frac{\sigma}{2\varepsilon_0}\left(1 + \frac{1}{\sqrt{1+a^2/z^2}}\right)\vec{e}_z & \text{si } z < 0 \end{cases}}$$

$$\lim_{z\to 0}\left(E(z>0) - E(z<0)\right) = \sigma/\varepsilon_0 .$$

$$\vec{E} = -\vec{\nabla}\phi \rightarrow \partial_r\phi = 0 = \partial_\theta\phi$$

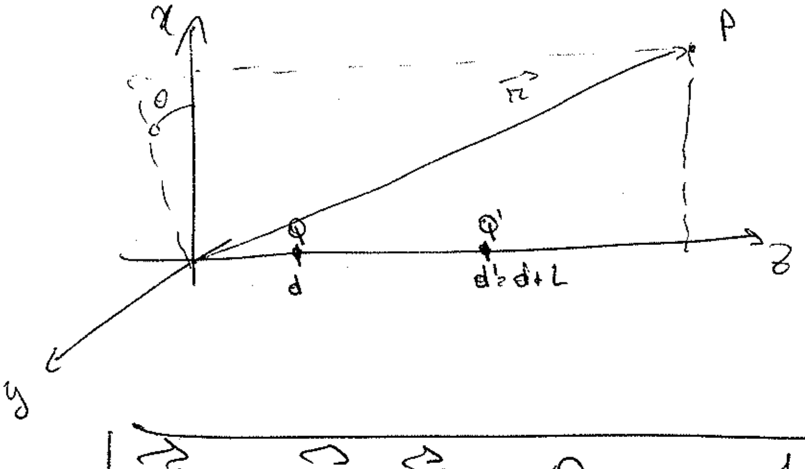
$$\phi(z) = -\int E(z)\,dz \quad \text{et} \quad \phi(0^+) = \phi(0^-)$$

$$\Rightarrow \boxed{\phi(z) = \begin{cases} \dfrac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+a^2} - z\right) + K & \text{si } z \geq 0 \\[2ex] \dfrac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+a^2} + z\right) + K & \text{si } z \leq 0 \end{cases}}$$

$$\lim_{z \to \pm\infty} \phi(z) = K = 0$$

$$\hookrightarrow \boxed{\phi(z) = \begin{cases} \dfrac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+a^2} - z\right) & \text{si } z \geq 0 \\[2ex] \dfrac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+a^2} + z\right) & \text{si } z \leq 0 \end{cases}}$$

Exo 6



$$\begin{cases} \vec{QP} = \vec{r} - d\,\vec{e_z} \\ \vec{Q'P} = \vec{r} - (d+L)\,\vec{e_z} \end{cases}$$

$$\vec{E}^{(1)} = \frac{Q^{(1)}}{4\pi\epsilon_0}\frac{\vec{Q^{(1)}P}}{|\vec{Q^{(1)}P}|^3}$$

$$\Rightarrow \boxed{\begin{aligned} \vec{E}_{TOT} = \vec{E} + \vec{E}' = {} & \frac{Q}{4\pi\epsilon_0}\frac{1}{(x^2+y^2+(z-d)^2)^{3/2}}(x, y, z-d) \\ & + \frac{Q'}{4\pi\epsilon_0}\frac{1}{(x^2+y^2+(z-d-L)^2)^{3/2}}(x, y, z-d-L) \end{aligned}}$$

$$\phi = -\int dx\, E_x = -\int dy\, E_y = -\int dz\, E_z$$

$$\boxed{\phi = \frac{Q}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{Q'}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+y^2+(z-d-L)^2}}}$$

$$\phi(x,y,z) \equiv 0$$

$$\Leftrightarrow x^2(\underbrace{Q'^2 - Q^2}_{\equiv \Delta Q^2}) + y^2(Q'^2 - Q^2) + z^2(Q'^2 - Q^2)$$
$$- 2z(\cancel{d}Q'^2 - \cancel{d}Q^2) = d'^2 Q^2 - d^2 Q'^2$$

$$\Leftrightarrow x^2 \Delta Q^2 + y^2 \Delta Q^2 + (z-d)^2 \Delta Q^2 = Q^2 L^2$$
$$- 2Q^2 L(z-d)$$

$$\begin{cases} X \equiv x\,\Delta Q \\ Y \equiv y\,\Delta Q \\ Z \equiv (z-d)\,\Delta Q \end{cases}$$

$$\Rightarrow X^2 + Y^2 + Z^2 + 2Z \cdot \frac{LQ^2}{\Delta Q} = Q^2 L^2 + \left(\frac{LQ^2}{\Delta Q}\right)^2$$
$$+ \left(\frac{LQ^2}{\Delta Q}\right)^2$$

$$\Leftrightarrow X^2 + Y^2 + \left(Z + L\frac{Q^2}{\Delta Q}\right)^2 = Q^2L^2\overbrace{\left(1 + \frac{Q^2}{\Delta Q^2}\right)}^{Q'^2/\Delta Q^2}$$

$\tilde{X} \equiv X$ || $\tilde{Z} \equiv Z + L\, Q^2/\Delta Q$

$\tilde{Y} \equiv Y$ || $R \equiv L\dfrac{QQ'}{\Delta Q}$ $\rightarrow$ $\boxed{\tilde{X}^2 + \tilde{Y}^2 + \tilde{Z}^2 = R^2}$ $\rightarrow$ Sphère

$\boxed{\text{Eno 11}}$

$$D(x,\alpha) = \frac{1}{\alpha\sqrt{2\pi}} e^{-x^2/2\alpha^2}$$

(i) $$\int_{-\infty}^{+\infty} dx\, D(x,\alpha) = 1$$

(ii) $$\int_{-\infty}^{+\infty} dx\, f(x) D(x,\alpha) \xrightarrow{\alpha\to 0} f(0)$$

Dem:

(i) $$\int_{-\infty}^{+\infty} dx\, \frac{e^{-x^2/2\alpha^2}}{\alpha\sqrt{2\pi}} = \frac{1}{\alpha\sqrt{2\pi}} \underbrace{\int_{-\infty}^{+\infty} dx\, e^{-x^2/2\alpha^2}}_{\equiv I(\alpha)}$$

$$I(\alpha)^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{x^2+y^2}{2\alpha^2}} dx\,dy = \int_0^{\infty}\int_0^{2\pi} r\, e^{-r^2/2\alpha^2} dr\, d\theta$$

$$= \dots = 2\pi\alpha^2 \Rightarrow \boxed{I(\alpha) = \alpha\sqrt{2\pi}}$$

$$\Rightarrow \int_{-\infty}^{+\infty} dx\, \frac{e^{-x^2/2\alpha^2}}{\alpha\sqrt{2\pi}} = 1 \qquad \square$$

(ii) $$\lim_{\alpha\to 0} \frac{1}{\alpha\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-\frac{(x-x_0)^2}{2\alpha^2}} dx = ?$$

$$\frac{x - x_0^*}{\sqrt{2}\cdot\alpha} \equiv X \Rightarrow \frac{dx}{\sqrt{2}\alpha} = dX$$

$$\Rightarrow \lim_{\alpha\to 0} \frac{1}{\cancel{\alpha}\sqrt{2\pi}} \cancel{\sqrt{2}}\cdot\cancel{\alpha} \int_{-\infty}^{+\infty} f(x_0+\alpha\sqrt{2}\cdot X)\, e^{-X^2} dX$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} f(x_0)\, e^{-X^2} dX = \frac{1}{\sqrt{\pi}} f(x_0) \underbrace{\int_{-\infty}^{+\infty} e^{-X^2} dX}_{=\Gamma(1/2) = \sqrt{\pi}}$$

$$= f(x_0)$$

$\square$

$\boxed{\text{Exo 12}}$

a) $\rho(r) = C\,\delta(r-R)$

$$Q = \int d^3r\, \rho(r) = \int_0^{\infty} dr \int_0^{2\pi} d\theta \int_0^{\pi} d\phi\, (r^2 \sin\theta \cdot C \cdot \delta(r-R))$$

$$= 4\pi R^2 C \longrightarrow C = \frac{Q}{4\pi R^2}$$

$$\longrightarrow \boxed{\rho(r) = \frac{Q}{4\pi R^2}\delta(r-R)}$$

b) $\rho(p) = C\cdot\delta(p-b)$

$$Q = \int_0^{h} dz \int_0^{2\pi} d\theta \underbrace{\int_0^{\infty} dp\, p\, C\,\delta(p-b)}_{= C\,2\pi b} = d \cdot \cancel{h}$$

$$\Rightarrow C = \frac{d}{2\pi b}$$

$$\longrightarrow \boxed{\rho(p) = \frac{d}{2\pi b}\delta(r-b)}$$

c) $\rho(r) = C\,\delta(z)\,\Theta(r-R)$

$$\Phi = \underbrace{\int_{-\infty}^{+\infty} dz \int_0^{\infty} dr \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \, r \cdot C \cdot \delta(z)}_{=1}\,\Theta(r-R)$$

$$= 2\pi C \int_0^{\infty} dr\, r\, \Theta(r-R) = 2\pi C \int_0^{R} dr\, r = 2\pi R \cdot C$$

$$\to \boxed{\rho(r) = \frac{\Phi}{2\pi R \cdot} \delta(z)\,\Theta(r-R)}$$

b) $\rho(r) = C \cdot \delta(\theta - \pi/2)\,\Theta(r-R)$

$$\Phi = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\infty} dr \; C \cdot \delta(\theta-\pi/2)\,\Theta(r-R)\, r^2 \sin\theta$$

$$= \frac{2}{3}\pi R^3 C$$

$$\Rightarrow \boxed{\rho(r) = \frac{3\Phi}{2\pi R^3} \delta(\theta - \pi/2)\,\Theta(r-R)}$$

$\boxed{\text{Exo 13}}$

Devoir

Exo 14 ⑧

$$\begin{cases} \phi(r) = \dfrac{q}{4\pi\epsilon_0}\,\dfrac{e^{-\alpha r}}{r}\left(1+\dfrac{\alpha r}{2}\right) \\ \Delta\phi = -\rho/\epsilon_0 \end{cases}$$

a)

$$\rightarrow \quad \rho(r) = -\epsilon_0\,\Delta\phi$$

$$= -\epsilon_0\,\frac{1}{r^2}\,\partial_r\left(r^2\,\partial_r\,\phi\right)$$

$$= \dots$$

$$\boxed{\rho(r) = -\frac{q}{8\pi}\,\alpha^3\,e^{-\alpha r}}$$

b) $\rho(r) = C\cdot\delta(r) = -\epsilon_0\,\Delta\phi$ proche de 0.

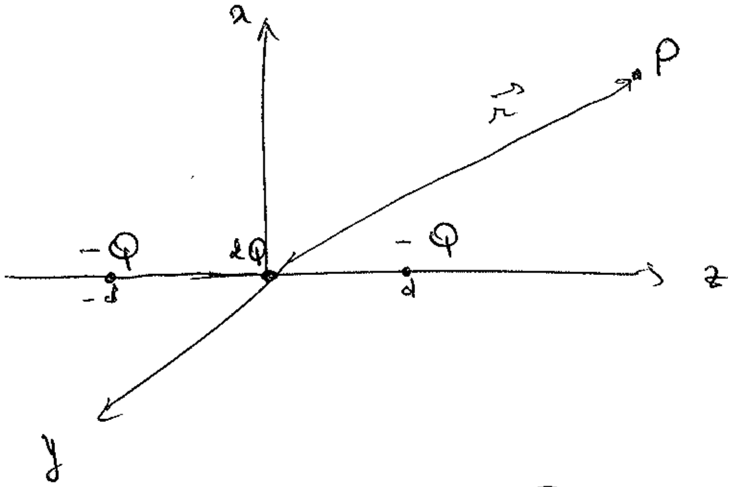
$$\phi(r\to 0) \simeq \frac{q}{4\pi\epsilon_0}\,\frac{1}{r}$$

$$\Leftrightarrow \quad C\delta(r) \simeq \frac{-q}{4\pi}\,\Delta\left(\frac{1}{r}\right)$$

$$= +\frac{q}{\cancel{4\pi}}\left(+\cancel{4\pi}\,\delta(r)\right) = q\,\delta(r) \Rightarrow C = q$$

$$\rightarrow \boxed{\rho(r) = q\,\delta(r)}$$

$\boxed{\text{Ex no 15}}$



$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\vec{r}-\vec{r}_i|}$$

$$\rightarrow \phi(\vec{r}) = \frac{Q}{4\pi\epsilon_0}\left[\frac{-1}{|\vec{r}-\vec{r}_1|} + \frac{2}{|\vec{r}-\vec{0}|} + \frac{-1}{|\vec{r}-\vec{r}_2|}\right]$$

$$\boxed{\phi(x,y,z) = \frac{Q}{4\pi\epsilon_0}\left[\frac{2}{\sqrt{x^2+y^2+z^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z-d)^2}}\right]}$$

$$\frac{1}{|\vec{r}-\vec{x}|} \simeq \frac{1}{|\vec{r}|} + \frac{\vec{r}\cdot\vec{x}}{|\vec{r}|^3} + \frac{1}{2}\,\frac{3(\vec{r}\cdot\vec{x})^2 - |\vec{r}|^2|\vec{x}|^2}{|\vec{r}|^5} + O(|\vec{r}|^{-4})$$

$$\rightarrow \phi(\vec{r}) \simeq \frac{Q}{4\pi\epsilon_0}\left[\cancel{\frac{2}{|\vec{r}|}} - \cancel{\frac{1}{|\vec{r}|}} - \cancel{\frac{\vec{r}\cdot\vec{r}_1}{|\vec{r}|^3}} - \frac{1}{2}\,\frac{3(\vec{r}\cdot\vec{r}_1)^2 - |\vec{r}|^2|\vec{r}_1|^2}{|\vec{r}|^5}\right.$$
$$\left. - \cancel{\frac{1}{|\vec{r}|}} + \cancel{\frac{\vec{r}\cdot\vec{r}_1}{|\vec{r}|^3}} - \frac{1}{2}\,\frac{3(\vec{r}\cdot\vec{r}_1)^2 - |\vec{r}|^2|\vec{r}_1|^2}{|\vec{r}|^5}\right]$$

$|\vec{r}|^2 = r^2$ | $\vec{r}\cdot\vec{r}_1 = rd\cos\theta$

$|\vec{r}_1|^2 = d^2$

$$\rightarrow \boxed{\phi(\vec{r}) = -\frac{Q}{4\pi\epsilon_0}\,\frac{d^2}{r^3}\left(3\cos^2\theta - 1\right)}$$

$\boxed{\text{Es. 016}}$

a) $\rho(r) = \begin{cases} 3Q/4\pi R^3 & 0 \le r \le R \\ 0 & R < r \end{cases}$

$$|\vec{E}| = \frac{1}{4\pi\epsilon_0 r^2} \int dV \rho(r) = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{1}{r^2} Q\left(\frac{r}{R}\right)^3 & r \le R \\ \frac{1}{4\pi\epsilon_0}\frac{1}{r^2} Q & r > R \end{cases}$$

$$W = \frac{1}{2}\epsilon_0 \int dV |\vec{E}(r)|^2$$
$$= \frac{1}{2}\epsilon_0 \underbrace{\int d\Omega}_{4\pi} \int_0^\infty dr\, |E(r)|^2$$
$$= \frac{1}{2} \cancel{4\pi}\epsilon_0 \cdot \frac{Q^2}{(4\pi\epsilon_0)^2} \left\{ \int_0^R \frac{r^4}{R^6} dr + \int_R^\infty \frac{1}{r^2} dr \right\}$$
$$\Rightarrow \boxed{W_a = \frac{3Q^2}{20\pi R\epsilon_0}}$$

b) $\sigma(r) = \begin{cases} \frac{Q}{4\pi R^2} & r = R \\ 0 & r \neq R \end{cases} \rightarrow E(r) = \begin{cases} 0 & r < R \\ \frac{\sigma}{\epsilon_0}\left(\frac{R}{r}\right)^2 & r \ge R \end{cases}$

$$W(r) = \frac{1}{2}\epsilon_0 \left\{ \int_0^R dr\, r^2 \cdot |0|^2 + \int_R^\infty dr\, \cancel{r^2} \frac{\sigma^2}{\epsilon_0^{\cancel{2}}} \frac{R^4}{r^{\cancel{4}2}} \right\} \underbrace{\int d\Omega}_{4\pi}$$
$$= 2\pi \frac{\sigma^2}{\epsilon_0} R^4 \left(\frac{-1}{\infty} - \frac{-1}{R}\right) = \boxed{2\pi R^3 \sigma^2/\epsilon_0 = W}$$

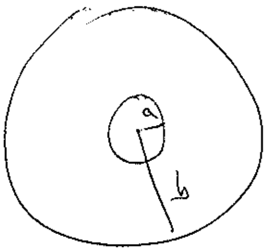
$$\Rightarrow \quad W = \frac{\cancel{4\pi} \cancel{R^3}}{\epsilon_0} \cdot \frac{Q^2}{\cancel{16}\pi^{\cancel{2}} R^{\cancel{4}}} \longrightarrow \boxed{W_b = \frac{Q^2}{8\pi R \epsilon_0}}$$

$$\frac{W_a}{W_b} = \frac{3\cancel{Q^2}/20\cancel{\pi}\cancel{R}\cancel{\epsilon_0}}{\cancel{Q^2}/8\cancel{\pi}\cancel{R}\cancel{\epsilon_0}} = \frac{3\cdot 8}{20} = \frac{3\cdot 4}{5} = 12/5 .$$

c) $W_a = \mathcal{E}_e \Leftrightarrow \dfrac{3Q^2}{20\pi R_e \epsilon_0} = m_e c^2$

$$\longrightarrow \boxed{R_e = 1{,}7 \cdot 10^{-15}\,m}$$

Exo 17



$$\Delta\phi = 0 \Leftrightarrow \frac{1}{r^2}\partial_r\left(r^2 \partial_r \phi\right) = 0$$

$$\hookrightarrow \boxed{\phi(r) = -\frac{K_1}{r} + K_2}$$

<u>$r \geq b$</u>

$$\begin{cases} \phi(\infty) = 0 \\ \phi(b) = \phi_b \end{cases} \longrightarrow \begin{cases} K_2 = 0 \\ K_1 = -b\,\phi_b \end{cases}$$

<u>$a \leq r \leq b$</u>

$$\begin{cases} \phi(b) = \phi_b \\ \phi(a) = \phi_a \end{cases} \longrightarrow \begin{cases} -\dfrac{K_1}{b} + K_2 = \phi_b \\ -\dfrac{K_1}{a} + K_2 = \phi_a \end{cases}$$

$$\longrightarrow \begin{cases} K_1 = (\phi_b - \phi_a)\dfrac{ab}{b-a} \\ K_2 = \dfrac{b\phi_b - a\phi_a}{b-a} \end{cases}$$

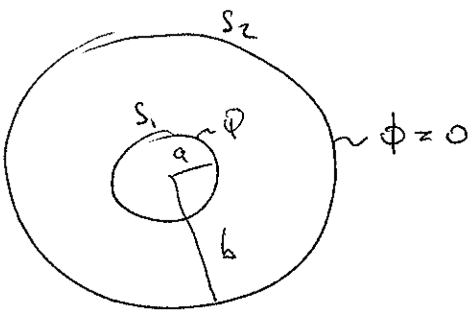
$r \leq a$

$$\begin{cases} \phi(a) = \phi_a \\ \phi(0) \neq \infty \end{cases} \longrightarrow \begin{cases} -\dfrac{K_1}{a} + K_2 = \phi_a \\ -\infty K_1 + K_2 \neq \infty \end{cases}$$

$$\longrightarrow \begin{cases} K_1 = 0 \\ K_2 = \phi_a \end{cases}$$

$$\Rightarrow \boxed{\phi(r) = \begin{cases} \phi_a & r \leq a \\ -\dfrac{1}{r}\dfrac{(\phi_b - \phi_a)ab}{b-a} + \dfrac{b\phi_b - a\phi_a}{b-a} & a \leq r \leq b \\ \dfrac{b}{r}\phi_b & b \leq r \end{cases}}$$

Exo 19



$$Q = 4\pi a^2 \cdot \sigma$$

$$E(r) = \begin{cases} 0 & \text{si } r \leq a \\ \dfrac{\sigma}{\epsilon_0}\left(\dfrac{a}{r}\right)^2 & \text{si } a < r < b \end{cases}$$

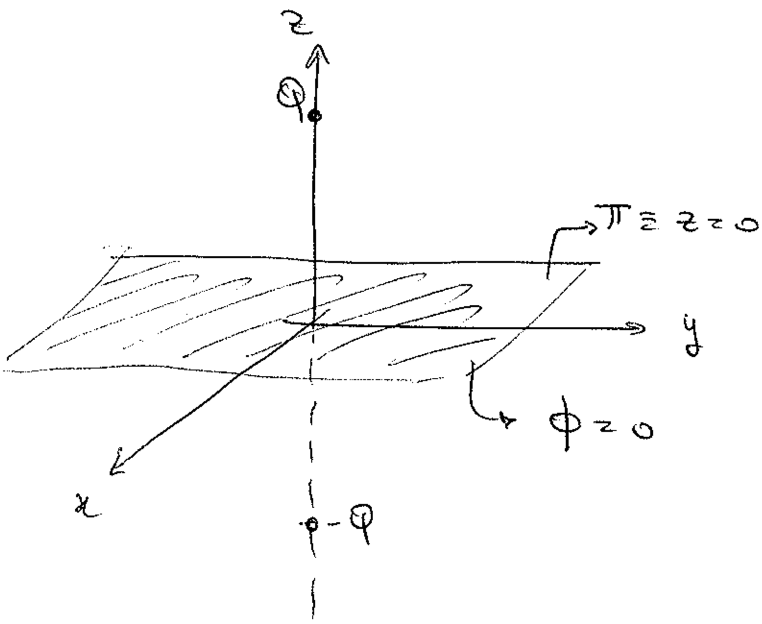
$$\phi(r) = -\int E(r)\,dr$$

$$= -\frac{\sigma}{\epsilon_0} a^2 \left(\frac{-1}{r}\right) + K$$

$$\phi(b) = 0 \longrightarrow K = -\frac{\sigma}{\epsilon_0} a^2/b$$

$$\Rightarrow \boxed{\phi(a) = \frac{\sigma a^2}{\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)}$$

$\boxed{\text{Ex 020}}$



a) $\phi(r) = \dfrac{Q}{4\pi\epsilon_0}\left[\dfrac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \dfrac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$

b) $\vec{E} = -\vec{\nabla}\phi$ et $\vec{E}\cdot\vec{n}_\pi\Big|_{z=0} = \dfrac{\sigma}{\epsilon_0}$

$\Rightarrow \sigma = -\epsilon_0\left(\vec{\nabla}\phi\cdot\vec{n}_\pi\right)_{z=0} = -\epsilon_0\,\partial_z\phi\Big|_{z=0}$

$$\boxed{\sigma = -\frac{Qd}{2\pi}\,\frac{1}{(x^2+y^2+d^2)^{3/2}}}$$

$$Q_\pi = \int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\;\sigma(x,y)$$

$$= -\frac{Qd}{2\pi}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\;\frac{1}{(x^2+y^2+d^2)^{3/2}} = -\frac{Qd}{\cancel{2\pi}}\int_0^{\infty}dr\int_0^{2\pi}\cancel{d\theta}\,\frac{r}{(r^2+d^2)^{3/2}}$$

$= -Q \longrightarrow \boxed{Q_\pi = -Q}$

c) $\vec{F}_{\varphi} = \varphi \vec{E}(0,0,d) = -\dfrac{\varphi^2}{16\pi d^2 \epsilon_0} \vec{u}_z$

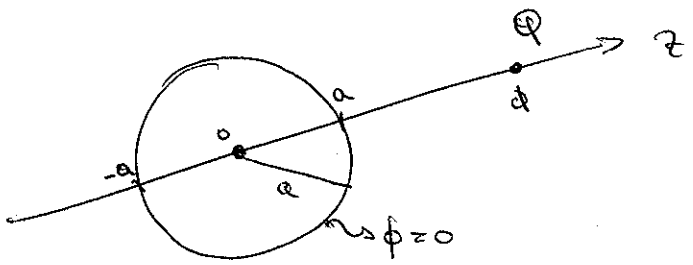
d) $W = \displaystyle\int_{C_\infty} \vec{F}_{\varphi} \cdot d\vec{x} = -\int_{(0,0,\infty)}^{(0,0,d)} \vec{\nabla}\psi \cdot d\vec{x} \cdot \varphi$

$$= -\varphi \int_{\infty}^{d} \partial_z \psi \, dz = -\varphi \left[\psi(d) - \underbrace{\psi(\infty)}_{=0} \right]$$

$$= -\varphi \cdot \frac{\varphi}{4\pi\epsilon_0 (2d)}$$

$$\to \boxed{W = -\frac{\varphi^2}{8\pi d \epsilon_0}}$$

Exo 21



$$\phi(x) = \frac{1}{4\pi\epsilon_0} \left[\frac{\varphi}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{\varphi'}{x^2+y^2+(z-d')^2} \right]$$

$\phi(a) = 0 = \phi(-a)$

$\to d' =$ a^2/d

d → Trivial.

$$\Rightarrow \varphi' = -\varphi\, a/d$$

$$\Rightarrow \boxed{\phi(x,y,z) = \frac{\varphi}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+\left(z-\frac{a^2}{d}\right)^2}}\right]}$$

$$\sigma(\theta) = \epsilon_0\left(\vec{E}\cdot\vec{u}_r\right)\Big|_{r=a} = -\epsilon_0\left(\vec{\nabla}\phi\cdot\vec{u}_r\right)_{r=a}$$

$$= -\epsilon_0\left(\partial_r\phi\Big|_{r=a}\right)$$

$$\cancel{= -\frac{\varphi}{4\pi}\left[\frac{-a+d\cos\theta}{(a^2+d^2-2ad\cos\theta)^{3/2}} + a\right.}$$

$$= -\frac{\varphi}{4\pi a^2}(d-a)\,\frac{a^2+d^2+ad(1-\cos\theta)}{(a^2+d^2-2ad\cos\theta)^{3/2}}$$

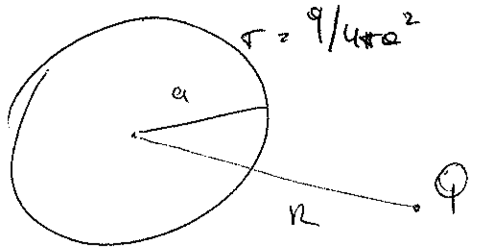
$$\varphi_{ind} = \int dS\,\sigma(\theta) = a^2\int_0^{\pi} d\theta\int_0^{2\pi} d\phi\,\sin\theta\,\sigma(\theta)$$

$$= 2\pi a^2\int_0^{\pi} d\theta\,\sin\theta\,\sigma(\theta).$$

$$= 2\pi a^2\cdot\frac{-\varphi}{2\pi a^2} = -\varphi$$

$$\longrightarrow \boxed{\varphi_{ind} = -\varphi}$$

Exo 22



① On fixe le potentiel à 0 s/ la sphère.

$\Rightarrow \exists\, q'$ tq $\phi|_{\text{sphère}} = 0$ (cf. Exo 21)

② On rajoute le potentiel créé p/ la vraie sphère :

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\vec{r}-\vec{d}|} + \frac{q'}{|\vec{r}-\vec{d}'|} + \frac{q-q'}{|\vec{r}|}\right]$$

↳ Potentiel dû à la charge nette s/ la surface de la sphère.

Exo 23

$\underline{n=1}$: $\partial_x^2\, G(x,x') = \delta(x-x') \Rightarrow G(x,x') = \frac{1}{2}|x|$ (Cours)

$\underline{n=2}$: $\left(\partial_r^2 + \frac{1}{r}\partial_r\right) G = 0 \Rightarrow G(r) = A\ln(r) + \cancel{B}$

fixons à 0.

$$1 = \iint_S \delta(r)\, dS = \iint_S \vec{\nabla}^2 G\, dS = \oint_{\partial S} \vec{\nabla} G\cdot\vec{n}\, dl = \oint_{\partial S} \frac{A}{r}\, r\, d\theta = 2\pi A$$

$$\longrightarrow \quad G(r) = \frac{1}{2\pi}\ln r$$

$\underline{n \geq 3:}$ Soit $G(r) = A \frac{1}{r^{n-2}}$

$$1 = \int_{V_n} \delta(x)\, d^n x = \int_{V_n} \vec{\nabla}^2 G\, d^n x = \oint_{\partial V_n = S_{n-1}} \vec{\nabla}\left(\frac{A}{r^{n-2}}\right) \cdot \vec{dS}$$

$$= A(-)(n-2) \int \frac{d\Omega_n}{r^{n-1}}$$

$$= A(-)(n-2) \frac{2\pi^{n/2} r^{n-1}}{\Gamma(n/2)\, r^{n-1}} \qquad \left(\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \right)$$

$$\Rightarrow \boxed{G(r) = -\frac{\Gamma(n/2)}{(n-2)\, 2\pi^{n-2}} \frac{1}{r^{n-2}}}$$

$\boxed{\text{Exo 24}}$

Cf. Cours:

Astuce: $\sin^3\left(\frac{\pi y}{b}\right) = \frac{3 \sin(\pi y/b) - \sin(3\pi y/b)}{4}$

$\phi(x,y,z) = U(x)V(y)W(z)$ (Séparation des variables)

$$\begin{cases} U(x) = A \sin(\alpha x) + B \cos(\alpha x) \\ V(y) = C \sin(\beta y) + D \cos(\beta y) \\ W(z) = E \sinh(\gamma z) + F \cosh(\gamma z) \end{cases}$$