Exo pour le devoir: 13 et 18

4 Electrostatics

1. Prove the following relations :

$$\begin{aligned}\vec\nabla(\vec a\cdot\vec b) &= (\vec a\cdot\vec\nabla)\vec b+(\vec b\cdot\vec\nabla)\vec a+\vec a\times(\vec\nabla\times\vec b)+\vec b\times(\vec\nabla\times\vec a)\\ \vec\nabla\cdot(\vec a\times\vec b) &= \vec b\cdot(\vec\nabla\times\vec a)-\vec a\cdot(\vec\nabla\times\vec b)\\ \vec\nabla\times(\vec a\times\vec b) &= \vec a(\vec\nabla\cdot\vec b)-\vec b(\vec\nabla\cdot\vec a)+(\vec b\cdot\vec\nabla)\vec a-(\vec a\cdot\vec\nabla)\vec b\end{aligned}$$

2. Prove that for any scalar function $\phi(x,y,z)$ the relation

$$\vec\nabla\times(\vec\nabla\phi)=0$$

holds. Show that for any vector field $\vec A(x,y,z)$ the following relation is true :

$$\vec\nabla\cdot(\vec\nabla\times\vec A)=0.$$

3. Compute the Laplacian $\Delta=\vec\nabla\cdot\vec\nabla$ on a scalar function ϕ in spherical coordinates.
4. Consider the following two vector fields
   i) $\vec A(x,y,z)=A_0(yz+yz^2,\ xz+xz^2,\ xy+2xyz)$
   ii) $\vec B(x,y,z)=B_0(x^2z,\ y^2z,\ xy^2)$
   where A_0 and B_0 are arbitrary (dimensionful) constants. Which of these vector fields can represent an electric field? For that field compute the resulting charge distribution and find a suitable electric potential.
5. Let the electric field be parallel to the x-axis in a given region of space,

$$\vec E=(E(x,y,z),0,0).$$

   a) Show that in that region the field cannot depend on y and z.
   b) Show that if there are no charges in this region, the field is independent of x as well.
6. Calculate the electric field and potential produced by a homogeneous charge distributed uniformly with the linear density λ along the z-axis.
7. Calculate the electric field and potential produced by the following charge distributions :
   a) $\rho(r)=\rho_0$ for $r<R$ and zero otherwise
   b) $\rho(r)=C/r$ for $r<R$ and zero otherwise
   c) spherical shell of the radius R and the surface charge density σ.
8. A perfect conductor has a cavity. Show that if a point charge q is placed inside the cavity, the charge $-q$ is induced on the internal surface of the cavity.
9. A uniformly charged disk of the radius a is placed at the origin in the plane xy. Calculate the electric field and potential produced by this disk at the points $(0,0,z)$ for positive and negative z. Calculate the limits $z\to 0_+$ and $z\to 0_-$.
10. Calculate the electric potential produced by two charges Q and Q', placed at $(0,0,d)$ and $(0,0,d')$, respectively. Find the equation of the surface of zero potential. Show that this surface is a sphere.
11. Show that the function

$$D(x,\alpha)=\frac{1}{\alpha\sqrt{2\pi}}\exp\left(-\frac{x^2}{2\alpha^2}\right)$$

goes to the Dirac δ-function in the limit $\alpha\to 0$.

12. By using the Dirac δ-function in appropriate coordinates, express the following charge distributions as three-dimensional charge densities $\rho(x)$:
   a) In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R.
   b) In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius b.
   c) In cylindrical coordinates, a charge Q spread uniformly over a flat circular disc of negligible thickness and radius R.
   d) The same as part c), but using spherical coordinates.
13. Each of three charged spheres of radius R, one conducting, one having a uniform charge density within its volume, and one having a spherically symmetric charge distribution that varies with radius as r^n ($n > -3$), has a total charge Q. Use Gauss's theorem to obtain the electric fields both inside and outside each sphere. Sketch the behavior of the field as a function of radius for the first two spheres, and for the third with $n = +2, -2$.
14. The time-averaged potential of a neutral hydrogen atom is given by

$$\Phi(r) = \frac{q}{4\pi\epsilon_0}\frac{\mathrm{e}^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right),$$

   where q is the magnitude of the electron charge, and $\alpha^{-1} = a_0/2$, $a_0 = 0.529 \times 10^{-10}$ m being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret the result physically.
15. Calculate the electrostatic potential produced by three point charges : $-Q$ in $(0,0,d)$, $2Q$ in $(0,0,0)$ and $-Q$ in $(0,0,-d)$. Calculate the multipole expansion up to and including the quadrupole term. Compare the exact result with the first non-zero term of the multipole expansion.
16. Calculate the electrostatic energy of the following charge distribution :

$$\rho(r) = \begin{cases} \frac{3Q}{4\pi R^3} & 0 \le r \le R \\ 0 & r > R \end{cases}$$

$$\sigma = \frac{Q}{4\pi R^2} \qquad \text{on the sphere of radius } R$$

   If one describes the electron as the uniformly charged sphere, one may derive the classical "electron radius" r_e by equating the electrostatic energy to the rest energy of the electron $c^2 m_e$. What is the value of r_e thus obtained ?
17. Calculate everywhere in space the solution to the Laplace equation that takes given values ϕ_a and ϕ_b on two concentric spheres of radii a and b, $b > a$, and is zero at infinity.
18. Calculate everywhere in space the solution to the Laplace equation that takes given values ϕ_a and ϕ_b on two concentric cylinders of radii a and b, $b > a$, and is zero at infinity.
19. Let a conducting sphere S_1 of radius a carrying charge Q be placed in the middle of a spherical cavity S_2 of radius b in a perfect conductor held at zero potential. Find the potential on the sphere S_1.
20. Let a charge Q be placed at the point $(0,0,d)$ above a conducting plane $z = 0$ held at zero potential. a) Find the potential everywhere in space. b) Calculate the density of charge and the total charge induced on the plane. c) Calculate the force acting on the charge Q. d) Calculate the work needed to bring this charge from infinity to the point $(0,0,d)$.

21. Let Q be a point charge placed outside of a conducting sphere of radius a held at zero potential. a) Calculate the distribution of charge induced on the sphere. b) Calculate the total charge of the sphere. The same questions for the charge placed inside the sphere.
22. Calculate the potential produced by a point charge Q placed at a distance R from the center of a conducting sphere of radius a having total charge q.
23. Consider the analog of the Poisson equation in the space with different number of dimensions $n = 1, 2, 4, ...,$

$$\left(\frac{\partial^2}{x_1^2} + \frac{\partial^2}{x_2^2} + \ldots + \frac{\partial^2}{x_n^2} \right) \Phi \equiv \Delta^{(n)} \Phi = \rho(x).$$

Find the Green's function of the corresponding Laplace equation, i.e., the function $G(x, x')$ satisfying the equation

$$\Delta^{(n)} G(x, x') = \delta^n(x - x').$$

24. Calculate the electrostatic potential inside a rectangular box of dimensions $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$, if all sides of the box are held at zero potential except the one at $z = c$ where the potential is $V(x, y) = V_0 \sin(\pi x/a) \sin^3(\pi y/b)$.
25. Calculate the electrostatic potential and electric field produced by an infinite cylinder of radius a with the axis along the direction of the axis z held at zero potential, placed in an external uniform field $\vec{E} = (E, 0, 0)$. *Hint : use the cylindrical coordinates ; outside of the cylinder solve the Laplace equation by separation of variables.*