Formalisme des quadri-vecteurs
Corrigés

Exo 1

$$\begin{cases} x_\mu = A_\mu{}^\nu x'_\nu \\ x^\mu = \Lambda^\mu{}_\nu \, x'^\nu \end{cases}$$

$\rightarrow \; x_\mu x^\mu = x'_\mu x'^\mu$

$\Leftrightarrow \; (A_\mu{}^\alpha x'_\alpha)(\Lambda^\mu{}_\beta x'^\beta) = x'_\mu x'^\mu$

$\Leftrightarrow \; A_\mu{}^\alpha \Lambda^\mu{}_\beta \, x'_\alpha x'^\beta = x'_\mu x'^\mu = x'_\alpha x'^\alpha = \delta^\alpha_\beta \, x'_\alpha x'^\beta$

$\Leftrightarrow \; A_\mu{}^\alpha A^\mu_\beta = \delta^\alpha{}_\beta$

Or $\; \Lambda^\mu_\beta \, (\Lambda^{-1})^\beta{}_\nu = \delta^\mu{}_\nu$

$\Rightarrow \; A_\mu{}^\alpha \delta^\mu{}_\nu = \delta^\alpha_\beta \, (\Lambda^{-1})^\beta{}_\nu$

$\Rightarrow \; \delta^\mu{}_\nu A_\mu{}^\alpha = \delta^\alpha{}_\beta (\Lambda^{-1})^\beta{}_\nu$

$\Leftrightarrow \; \delta_\nu{}^\mu A_\mu{}^\alpha = \delta^\alpha_\beta (\Lambda^{-1})^\beta{}_\nu$

$\Leftrightarrow \; A_\nu{}^\alpha = \underbrace{\delta^\alpha_\beta (\Lambda^{-1})^\beta{}_\nu}_{\text{Sym.}} = (\Lambda^{-1})^\alpha{}_\nu$

$\Rightarrow \; A^\alpha{}_\nu = (\Lambda^{-1})^\alpha{}_\nu \longrightarrow \boxed{A = \Lambda^{-1}}$

$\Lambda^{-1}(\theta) = \Lambda(-\theta)$.

Exo 2

$$x_\mu y^\mu = x_\mu \eta^{\mu\nu} y_\nu = \eta^{\mu\nu} x_\mu y_\nu = \eta^{\nu\mu} x_\mu y_\nu = x^\nu y_\nu$$
$$= x^\mu y_\mu .$$

Exo 3

$$\begin{cases} ct' = \gamma\left(ct - \frac{v}{c} x\right) \\ x' = \gamma(x - vt) = 0 \Rightarrow x = vt \end{cases}$$

$$\Rightarrow ct' = \gamma\left(ct - \left(\frac{v}{c}\right)^2 ct\right) = \gamma \overbrace{(1-\beta^2)}^{\gamma^{-2}} ct$$
$$= \gamma^{-1} \cdot ct$$

$$ds^2 = (ct)^2 - x^2 = (ct)^2 - (vt)^2 = t^2(c^2 - v^2) > 0 .$$

$$\rightarrow x^\mu x_\mu > 0 .$$

Exo 4

$$\begin{cases} ct' = \gamma\left(ct - \frac{v}{c} x\right) = 0 \Rightarrow x = \frac{c^2 t}{v} \\ x' = \gamma(x - vt) \end{cases}$$

$$\Rightarrow x' = \gamma\left(\left(\frac{c}{v}\right)^2 vt - vt\right) = \gamma \underbrace{(\beta^{-2} - 1)}_{\gamma^{-2}} v \cdot t$$
$$= \frac{\gamma}{\beta^2}(1 - \beta^2) vt$$
$$= \gamma^{-1}\beta^{-2} vt$$

$$ds^2 = (ct)^2 - x^2 = (ct)^2 - (ct)^2 \cdot \frac{c^2}{v^2} = (ct)^2 \underbrace{\left(1 - \frac{1}{\beta^2}\right)}_{< 0}$$

$$\rightarrow x^\mu x_\mu < 0 .$$

Ex. 5

$x_1^\mu x_{1\mu} > 0 \Rightarrow x_1^\mu = (ct_1, \vec{0})$

$x_i^\mu \equiv (ct_i, \vec{x}_i)$

① $x_1^\mu x_{j\mu} = 0$

$\Leftrightarrow (ct_1)(ct_j) - \vec{0}\cdot\vec{x}_j = 0 \Rightarrow t_j = 0 \quad \forall j = 2,3,4$

$\hookrightarrow x_i^\mu = (0, \vec{x}_i) \quad (x_i^2 < 0)$

② $x_i^\mu x^j_\mu = \delta_i^j$

$\Leftrightarrow \vec{x}_i \cdot \vec{x}_j = \delta_{ij} \rightarrow \begin{cases} \vec{x}_2 = (1,0,0) \\ \vec{x}_3 = (0,1,0) \\ \vec{x}_4 = (0,0,1) \end{cases}$ p/ex.

Ex. 6

$x^\mu x_\mu = 0 \rightarrow x^\mu = (x^0, \pm x^0, 0, 0)$

$y_i^\mu = (y_i^0, y_i^1, y_i^2, y_i^3)$

ⓐ $x^\mu y_{\mu i} = x^0(y_i^0 \mp y_i^1) = 0 \Leftrightarrow \boxed{y_i^0 = \pm y_i^1}$

$\rightarrow y_i^\mu = (y_i^0, \pm y_i^0, y_i^2, y_i^3)$

$y_i^\mu y_{j\mu} = \cancel{y_i^0 y_{j0}} - \cancel{y_i^0 y_{0j}} - y_i^2 y_{2j} - y_i^3 y_{3j} = 0 \quad (i \neq j)$

$\Leftrightarrow y_i^2 y_{2j} = -y_i^3 y_{3j} \quad (i \neq j)$

P/exemple

$$\Rightarrow \begin{cases} y_1^\mu = (y_1^0, \pm y_1^0, 0, 0) \\ y_2^\mu = (y_2^0, \pm y_2^0, y_2^2, 0) \\ y_3^\mu = (y_3^0, \pm y_3^0, 0, y_3^3) \end{cases}$$

$\boxed{\text{Exo 7}}$

$$x^\mu x_\mu = 0 \Rightarrow x^\mu = (x^0, \pm x^0)$$

$$\lambda x^\mu + \rho y^\mu = 0 \Leftrightarrow \lambda = \rho = 0$$

$$\Leftrightarrow y^\mu = (x^0, \mp x^0)$$

$\boxed{\text{Exo 8}}$

$$\begin{cases} x^\mu = (x^0, x^0, 0, 0) \\ y^\mu = (x^0, -x^0, 0, 0) \end{cases} \qquad z^\mu = (z^0, z^1, z^2, z^3)$$

$$z^\mu : \Rightarrow z^\mu x_\mu = z^\mu y_\mu = 0 \ \underline{\text{et}} \ \lambda z^\mu + \rho x^\mu + \sigma y^\mu = 0 \Leftrightarrow \lambda = \rho = \sigma = 0$$

$$\begin{cases} x^\mu z_\mu = 0 \Rightarrow z^0 x_0 - z^1 x_0 = 0 \\ y^\mu z_\mu = 0 \Rightarrow z^0 x_0 + z^1 x_0 = 0 \end{cases} \Rightarrow \begin{cases} z^0 - z^1 = 0 \\ z^0 + z^1 = 0 \end{cases}$$

$$\Leftrightarrow z^0 = z^1 = 0$$

$$\rightarrow z^\mu = (0, 0, z^2, z^3)$$

$$\Rightarrow \begin{cases} z^\mu = (0, 0, z^2, 0) & z^2 < 0 \\ \tilde{z}^\mu = (0, 0, 0, z^3) & \tilde{z}^2 < 0 \end{cases}$$

P/exemple

Exo 9

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \quad : \quad \Lambda = \begin{pmatrix} \mathrm{Ch}\,\theta & \mathrm{Sh}\,\theta & 0 & 0 \\ \mathrm{Sh}\,\theta & \mathrm{Ch}\,\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x^{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu} x'^{\nu} \quad : \quad \Lambda^{-1} = \begin{pmatrix} \mathrm{Ch}\,\theta & -\mathrm{Sh}\,\theta & 0 & 0 \\ -\mathrm{Sh}\,\theta & \mathrm{Ch}\,\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{Or, } \Lambda^{-1} = \begin{pmatrix} \mathrm{Ch}(-\theta) & \mathrm{Sh}(-\theta) & \\ \mathrm{Sh}(-\theta) & \mathrm{Ch}(-\theta) & O_2 \\ & O_2 & \mathbb{1}_2 \end{pmatrix}$$

$$\text{Car } \begin{cases} \mathrm{Ch}\,\theta = \mathrm{Ch}(-\theta) \\ \mathrm{Sh}\,\theta = -\mathrm{Sh}(-\theta) \end{cases}$$

$\Rightarrow$ Λ^{-1} correspond au boost de paramètre $(-\theta)$.

Exo 10

$$\epsilon_{\mu\nu\lambda\rho} = \eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\lambda\gamma}\eta_{\rho\delta}\,\epsilon^{\alpha\beta\gamma\delta}$$

$$\Rightarrow \epsilon_{0123} = \eta_{0\alpha}\eta_{0\beta}\eta_{0\gamma}\eta_{0\delta}\,\epsilon^{\alpha\beta\gamma\delta}$$

$$= \eta_{00}\eta_{11}\eta_{12}\eta_{13}\,\epsilon^{0123} = (-1)^3\,\epsilon^{0123} = -\underbrace{\epsilon^{0123}}_{=1}$$

$$\Rightarrow \boxed{\epsilon_{0123} = -1}$$

$\square$

$\boxed{\text{Exo } 11}$

$$\partial^{\mu} = \frac{\partial}{\partial x_{\mu}} \quad \text{et} \quad \partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$$

$$\partial^{\mu}\partial_{\mu} = \eta^{\mu\nu}\partial_{\nu}\partial_{\mu} = \eta^{\mu\nu}\frac{\partial}{\partial x_{\mu}}\frac{\partial}{\partial x_{\nu}}$$

$$= \frac{\partial}{\partial x_0}\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_3}$$

$$= \frac{1}{c^2}\partial_t^2 - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

$$= \frac{1}{c^2}\partial_t^2 - \vec{\nabla}^2$$

$$= \square$$ ("d'Alembertien").

$\boxed{\text{Exo } 12}$

Devoir.