Transformations de Lorentz
Corrigés

Exo 1

$$\left(\frac{1}{c^2}\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2\right)\phi(t,x,y,z) = 0$$

Où $\partial_\alpha \equiv \frac{\partial}{\partial \alpha}$.

ⓐ Transfo. de Galilée :
$$\begin{cases} t = t' \\ x = x' + v.t' \\ y = y' \\ z = z' \end{cases}$$

Invariance sous changement de coordonnées :

$$\phi(t,\vec{x}) = \phi'(t'(t), x'(t,x))$$

$$\Rightarrow \partial_t \phi = \partial_t \phi' = \frac{\partial \phi'}{\partial t'}\cdot\frac{\partial t'}{\partial t} + \frac{\partial \phi'}{\partial x'}\frac{\partial x'}{\partial t}$$

$$= \partial_{t'}\phi' - v\,\partial_{x'}\phi'$$

$$\Rightarrow \partial_t^2\phi = \partial_t\left(\partial_{t'}\phi' - v\partial_{x'}\phi'\right)$$

$$= \partial_{t'}(\partial_t\phi') - v\,\partial_{x'}(\partial_t\phi')$$

$$= \partial_{t'}^2\phi' - 2v\,\partial_{t'}\partial_{x'}\phi' + v^2\partial_{x'}^2\phi'$$

$$\Rightarrow \partial_x\phi = \partial_x\phi' = \frac{\partial\phi'}{\partial t'}\cancel{\frac{\partial t'}{\partial x}} + \frac{\partial\phi'}{\partial x'}\frac{\partial x'}{\partial x} = \partial_{x'}\phi'$$

$$\Rightarrow \partial_x^2\phi = \partial_{x'}^2\phi'$$

$\Rightarrow \partial_y^2 \phi = \partial_{y'}^2 \phi'$

$\Rightarrow \partial_z^2 \phi = \partial_{z'}^2 \phi'$

$$\Rightarrow \left(\frac{1}{c^2}\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2\right)\phi$$

$$= \left(\frac{1}{c^2}\partial_{t'}^2 - 2\frac{v}{c^2}\partial_{t'}\partial_{x'} + \frac{v^2}{c^2}\partial_{x'}^2 - \partial_{x'}^2 - \partial_{y'}^2 - \partial_{z'}^2\right)\phi'$$

$$\neq \left(\frac{1}{c^2}\partial_{t'}^2 - \partial_{x'}^2 - \partial_{y'}^2 - \partial_{z'}^2\right)\phi'$$

$\hookrightarrow$ Pas invariant!

(b) Transfo. de Lorentz : $\begin{cases} t' = \gamma\left(t - \frac{\beta}{c}x\right) \\ x' = \gamma(x - ct\beta) \\ y' = y \\ z' = z \end{cases}$

Invariance sous changement de coordonnées :

$$\phi(t,x) = \phi'(t'(x,t), x'(x,t))$$

$$\Rightarrow \partial_t \phi = \partial_t \phi' = \frac{\partial \phi'}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial \phi'}{\partial x'}\frac{\partial x'}{\partial t}$$

$$= \gamma\left(\partial_{t'}\phi' - v\partial_{x'}\phi'\right)$$

$$\Rightarrow \partial_t^2 \phi = \gamma^2\left(\partial_{t'}^2\phi' - 2v\partial_{t'}\partial_{x'}\phi' + v^2\partial_{x'}^2\phi'\right)$$

$$\Rightarrow \partial_x \phi = \partial_x \phi' = \frac{\partial \phi'}{\partial t'}\frac{\partial t'}{\partial x} + \frac{\partial \phi'}{\partial x'}\frac{\partial x'}{\partial x}$$

$$= \gamma\left(\partial_{x'}\phi' - \frac{\beta}{c}\partial_{t'}\phi'\right)$$

$$\Rightarrow \partial_x^2 \phi = \gamma^2 \left(\partial_{x'}^2 \phi' - 2\frac{\beta}{c} \partial_{t'} \partial_{x'} \phi' + \frac{\beta^2}{c^2} \partial_{t'}^2 \phi' \right)$$

$$\Rightarrow \frac{1}{c^2}\partial_t^2 - \partial_x^2 = \frac{\gamma^2}{c^2}\left(\partial_{t'}^2 - \cancel{2v\partial_{t'}\partial_{x'}} + v^2 \partial_{x'}^2 \right)$$

$$- \gamma^2 \left(\partial_{x'}^2 - \cancel{2\frac{\beta}{c}\partial_{t'}\partial_{x'}} + \frac{\beta^2}{c^2}\partial_{t'}^2 \right)$$

$$= \frac{\cancel{\gamma^2}}{c^2}\partial_{t'}^2 \left(1 - \beta^2 \right) - \cancel{\gamma^2} \partial_{x'}^2 \left(1 - \beta^2 \right)$$

$$= \frac{1}{c^2}\partial_{t'}^2 - \partial_{x'}^2$$

$$\Rightarrow \left(\frac{1}{c^2}\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2 \right) \phi$$

$$= \left(\frac{1}{c^2}\partial_{t'}^2 - \partial_{x'}^2 - \partial_{y'}^2 - \partial_{z'}^2 \right) \phi'$$

$\hookrightarrow$ Invariant !

$\boxed{\text{Exo 2}}$

Soit $\eta \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow (ct)^2 - x^2 = (ct \;\; x) \cdot \eta \cdot \begin{pmatrix} ct \\ x \end{pmatrix}$

Si $\begin{pmatrix} ct \\ x \end{pmatrix} = M \cdot \begin{pmatrix} ct' \\ x' \end{pmatrix} \Rightarrow (ct \;\; x) = (ct' \;\; x') \cdot M^t$

$$\Rightarrow (ct)^2 - x^2 = (ct' \;\; x') \cdot M^t \cdot \eta \cdot M \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

Invariance de Δs^2 $\rightarrow = (ct')^2 - x'^2 = (ct' \;\; x') \, \eta \begin{pmatrix} ct' \\ x' \end{pmatrix}$

$$\Rightarrow \quad M^t \eta M = \eta$$

Soit $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} A^2 - C^2 = 1 \\ AB - CD = 0 \\ B^2 - D^2 = -1 \end{cases}$$

Solutions: $\begin{cases} D = \pm A \\ C = \pm B \end{cases}$

$$\Rightarrow M = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \text{ ou } M = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}.$$

$\square$.

$\boxed{\text{Exo 3}}$

Boost 1: $\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} ch\theta_1 & sh\theta_1 \\ sh\theta_1 & ch\theta_1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$

Boost 2: $\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} ch\theta_2 & sh\theta_2 \\ sh\theta_2 & ch\theta_2 \end{pmatrix} \begin{pmatrix} ct'' \\ x'' \end{pmatrix}$

(53)

$$\Rightarrow \begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \mathrm{Ch}\theta_1 & \mathrm{Sh}\theta_1 \\ \mathrm{Sh}\theta_1 & \mathrm{Ch}\theta_1 \end{pmatrix} \begin{pmatrix} \mathrm{Ch}\theta_2 & \mathrm{Sh}\theta_2 \\ \mathrm{Sh}\theta_2 & \mathrm{Ch}\theta_2 \end{pmatrix} \begin{pmatrix} ct'' \\ x'' \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{Ch}\theta_1 \cdot \mathrm{Ch}\theta_2 + \mathrm{Sh}\theta_1 \mathrm{Sh}\theta_2 & \mathrm{Ch}\theta_1 \mathrm{Sh}\theta_2 + \mathrm{Sh}\theta_1 \mathrm{Ch}\theta_2 \\ \mathrm{Sh}\theta_1 \mathrm{Ch}\theta_2 + \mathrm{Ch}\theta_1 \mathrm{Sh}\theta_2 & \mathrm{Sh}\theta_1 \mathrm{Sh}\theta_2 + \mathrm{Ch}\theta_1 \mathrm{Ch}\theta_2 \end{pmatrix}$$

Or, $\mathrm{Ch}\theta_1 \mathrm{Ch}\theta_2 + \mathrm{Sh}\theta_1 \mathrm{Sh}\theta_2$

$$= \frac{1}{2}\left(e^{\theta_1} + e^{-\theta_1}\right)\frac{1}{2}\left(e^{\theta_2} + e^{-\theta_2}\right)$$

$$+ \frac{1}{2}\left(e^{\theta_1} - e^{-\theta_1}\right)\frac{1}{2}\left(e^{\theta_2} - e^{-\theta_2}\right)$$

$$= \frac{1}{4}\left\{ e^{\theta_1+\theta_2} + \cancel{e^{\theta_1-\theta_2}} + \cancel{e^{-\theta_1+\theta_2}} + e^{-\theta_1-\theta_2} + e^{\theta_1+\theta_2} - \cancel{e^{\theta_1-\theta_2}} - \cancel{e^{-\theta_1+\theta_2}} + e^{-\theta_1-\theta_2} \right\}$$

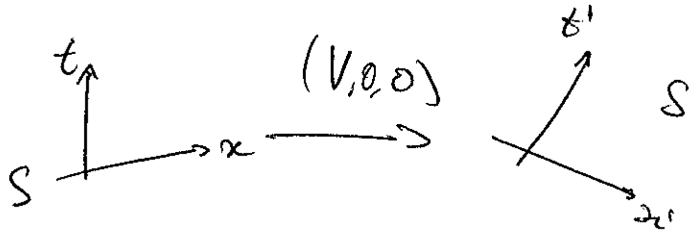
$$= \frac{1}{2}\left(e^{\theta_1+\theta_2} + e^{-(\theta_1+\theta_2)}\right) = \mathrm{Ch}(\theta_1+\theta_2).$$

Idem pour $\mathrm{Ch}\theta_1 \mathrm{Sh}\theta_2 + \mathrm{Sh}\theta_1 \mathrm{Ch}\theta_2$

$$= \mathrm{Sh}(\theta_1+\theta_2)$$

$\square$.

Exo 4



$$S : \begin{cases} t = 1\,\mu s \\ x = 3\,km \end{cases}$$

On impose la simultanéité dans le nouveau référentiel.

$$\begin{cases} ct' = \gamma(ct - \beta x) = 0 \\ x' = \gamma(x - \beta ct) \end{cases}$$

$$\Rightarrow \quad ct = \beta x \quad \Rightarrow \quad \beta = \frac{ct}{x} = 0{,}1$$

$$\Rightarrow \quad x' = \gamma(x - \beta ct) = \underline{2{,}98\ km}$$

$$\underline{(ct' ; x') = (0 ; 2{,}98\ km)}$$

Exo 5

$$\begin{cases} \Delta x' = L = 300\ m \\ \Delta t' = 0\ s \end{cases}$$

$$\frac{V}{c} = \beta = 10^{-3} \quad \rightarrow \quad \gamma = 1{,}0000005$$

$$\begin{cases} \Delta x = \gamma(\Delta x' + v\,\Delta t') = \gamma L = \underline{300{,}000\,15\ m} \\ \Delta t = \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right) = \gamma \frac{\beta}{c} L = 1{,}000\,000\,5 \cdot 10^{-9}\ s \end{cases}$$

$$\simeq \underline{1\ ns}$$

Exo 6

$$\begin{cases} \Delta x' = 10^3 \text{ m} \\ \Delta t' = 10^{-6} \text{ s} \end{cases}$$

$$\begin{cases} x_1 = \gamma(x_1' + v t_1') \\ t_1 = \gamma\left(t_1' + \frac{v}{c^2} x_1'\right) \end{cases} \quad \underline{\text{et}} \quad \begin{cases} x_2 = \gamma(x_2' + v t_2') \\ t_2 = \gamma\left(t_2' + \frac{v}{c^2} x_2'\right) \end{cases}$$

Avec, S' au repos et S en mouvement.

$$\begin{cases} x_2' - x_1' = 10^3 \text{ m} \\ t_2' - t_1' = 10^{-6} \text{ s} \end{cases}$$

$$\Rightarrow \quad t_2 - t_1 = \gamma\left[(t_2' - t_1') + \frac{v}{c^2}(x_2' - x_1')\right]$$

$= 0$

On impose la simultanéité dans S.

$$\Rightarrow \boxed{\frac{v}{c} = 0{,}3}$$

$\boxed{\text{Exo 7}}$

$$\begin{cases} ct' = \gamma(ct - \frac{v}{c}x) \\ x' = \gamma(x - vt) \end{cases} \rightarrow \begin{cases} cdt' = \gamma(cdt - \frac{v}{c}dx) \\ dx' = \gamma(dx - vdt) \end{cases}$$

Si $dt' = dt \Rightarrow cdt = \gamma(cdt - \frac{v}{c}dx)$

$$\Rightarrow \frac{dx}{dt} \equiv u = c\,\frac{\gamma - 1}{\gamma\beta} = c\,\frac{\text{th}}{}$$

$$u' = \frac{dx'}{dt'} = \frac{dx'}{dt} = \gamma(u - v)$$

$$= \gamma\left(c\,\frac{\gamma - 1}{\gamma\beta} - v\right)$$

$$= \dots$$

$$= c\,\frac{1-\gamma}{\gamma\beta} = -u$$

Soit S'' le référentiel de la particule : $(dx'' = 0)$

$$\begin{cases} cdt = \gamma_s\, cdt'' \\ dx = \gamma_s\, u\, dt'' \end{cases} \quad \text{et} \quad \begin{cases} cdt' = \gamma_{s'}\, cdt'' \\ dx' = \gamma_{s'}\, u'\, dt'' \end{cases}$$

$$\Rightarrow \frac{dt}{dt'} = \frac{\gamma_s}{\gamma_{s'}} = \frac{\sqrt{1 - u'^2/c^2}}{\sqrt{1 - u^2/c^2}} = \sqrt{\frac{1 - (-u)^2/c^2}{1 - u^2/c^2}} = 1$$

$$\Rightarrow \boxed{dt = dt'}.$$

L'intervalle de temps est le même car $\underline{\underline{\gamma = \gamma(u^2)}}$.

Exo 8

Position du μ-mesons dans son réf.

$$\begin{cases} \tau \simeq 2.10^{-6}s = t' \quad \text{et} \quad x' = 0 \\ h \simeq 10^4 m = x \end{cases}$$

$$\begin{cases} ct = \gamma(ct' + \frac{V}{c^2}x') \\ x = \gamma(x' + Vt') \end{cases}$$

$$\Rightarrow \begin{cases} t = \gamma\tau \\ h = \gamma V\tau \end{cases} \rightarrow V = \frac{h}{\sqrt{\tau^2 + h^2/c^2}} = \underline{0,998205\, c}$$

Exo 9

$$\begin{cases} L' = 100\ m \\ \ell = 120\ m \end{cases}$$

S : Navette

S' : Station

$$\begin{cases} \Delta x' = \gamma(\Delta x' \mp \beta c\Delta t') \\ c\Delta t' = \gamma(\Delta x'\beta \mp c\Delta t') = 0 \end{cases}$$

p/ simultanéité

$$\Rightarrow \ell\beta = c\Delta t \Rightarrow \boxed{\ell = \gamma \ell'}$$ (Contract° Lorentzienne)

Pour avoir $L' \geq \ell'$ $\Rightarrow$ $\boxed{V \geq 0,55\,c}$

$\geq \frac{\ell}{\gamma}$ $\Rightarrow$

$\Rightarrow \gamma \geq \ell/L'$

$\boxed{\text{Exo 10}}$ /

$\boxed{\text{Exo 11}}$

$$\begin{cases} ct = \gamma\left(ct' + \frac{v}{c}y'\right) \\ x = x' \\ y = \gamma\left(y' + \frac{v}{c}ct'\right) \\ z = z' \end{cases}$$

$$\begin{cases} ct' = \gamma\left(ct'' + \beta x''\right) \\ x' = \gamma\left(x'' + \beta ct''\right) \\ y' = y'' \\ z' = z'' \end{cases}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & 0 & \gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & \cancel{0} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct'' \\ x'' \\ y'' \\ z'' \end{pmatrix}$$

$$= \begin{pmatrix} \gamma^2 & \gamma^2\beta & \gamma\beta & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ \gamma^2\beta & \gamma^2\beta^2 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct'' \\ x'' \\ y'' \\ z'' \end{pmatrix}$$

$$\Rightarrow \begin{cases} c\,dt = \gamma^2 c\,dt'' + \gamma^2\beta\, dx'' + \gamma\beta\, dy'' \\ dx = \gamma\beta c\,dt'' + \gamma\, dx'' \\ dy = \gamma^2\beta\, c\,dt'' + \gamma^2\beta^2 dx'' + \gamma\, dy'' \\ dz = dz'' \end{cases}$$

$$u_x = \frac{dx}{dt} = \frac{\gamma\beta c\,dt'' + \gamma\, dx''}{\gamma^2 c\,dt'' + \gamma^2\beta\, dx'' + \gamma\beta\, dy''} \overset{\beta = V/c}{=} \frac{V}{\gamma + \frac{Vv}{c^2}}$$

$$u_y = \frac{dy}{dt} = \frac{\gamma^2\beta\, c\,dt'' + \gamma^2\beta^2 dx'' + \gamma\, dy''}{\gamma^2 c\,dt'' + \gamma^2\beta\, dx'' + \gamma\beta\, dy''} = \frac{\gamma V + v}{\gamma + \frac{Vv}{c^2}}$$

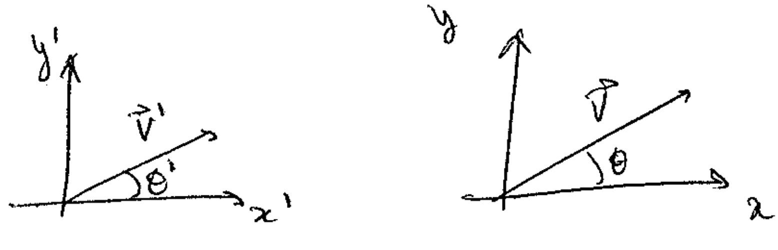
$$u_z = \frac{dz}{dt} = \frac{dz''}{\gamma^2 c\,dt'' + \gamma^2\beta\, dx'' + \gamma\beta\, dy''} = 0$$

$$\frac{dz''}{dt''} = u_z'' = 0.$$

$$\hookrightarrow \boxed{\vec{v} = \frac{1}{\gamma + \frac{Vv}{c^2}} \left(V, \gamma V + v, 0\right)}$$

Exo 12

$$\begin{cases} ct = \gamma\left(ct' + \frac{v}{c}x'\right) \\ x = \gamma\left(x' + \frac{v}{c}ct'\right) \\ y = y' \\ z = z' \end{cases}$$



$$\rightarrow \begin{cases} dx = \gamma\left(dx' + \frac{v}{c} c\,dt'\right) \\ dt = \gamma\left(dt' + \frac{v}{c^2} dx'\right) \\ dy = dy' \\ dz = dz' \end{cases}$$

$$\begin{cases} V_x' = V' \cos\theta' \\ V_y' = V' \sin\theta' \\ V_z' = 0 \end{cases}$$

$$V_x = \frac{dx}{dt} = \frac{dx' + \beta c\, dt'}{dt' + \frac{\beta}{c} dx'} = \frac{V_x' + V}{1 + \frac{V V_x'}{c^2}}$$

$$V_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{V}{c^2} dx'\right)} = \frac{V_y'}{\gamma\left(1 + \frac{V V_x'}{c^2}\right)}$$

$$\theta = \tan^{-1}\left[\frac{V_y}{V_x}\right] = \tan^{-1}\left[\frac{V_y'}{\gamma(V_x' + V)}\right]$$

$$\boxed{\theta = \tan^{-1}\left[\frac{V' \sin\theta'}{(V'\cos\theta' + V)\gamma}\right]}$$

Exo 13

$$\Delta\theta = \theta' - \theta = \theta' - \tan^{-1}\left[\frac{V' \sin\theta'}{(V'\cos\theta' + V)\gamma}\right]$$

Ici : $V' = c$ car lumière.

$$\Rightarrow \Delta\theta = \theta' - \tan^{-1}\left[\frac{\sin\theta'}{(\cos\theta' + V/c)\gamma}\right] \rightarrow \gamma = \gamma(\beta) \quad ⚠$$

$$\cancel{\Delta\theta \overset{\beta \ll 1}{\approx} \theta' - \tan^{-1}\left[\gamma^{-1} \tan\theta'\right] + O(\beta)}$$

$$\boxed{\Delta\theta \simeq \sin(\theta)\cdot\beta - \frac{1}{4}\sin(2\theta)\cdot\beta^2 + O(\beta^3)}$$

$\boxed{\text{Exo } 14}$

Devoir.