1 Lorentz transformations

1. Show that the wave equation
$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)\phi(t,x,y,z) = 0$$
is *not* invariant under Galilean transformations, but *is* invariant under Lorentz transformations.
2. Show that all linear transformations
$$\begin{pmatrix} ct \\ x \end{pmatrix} = M \cdot \begin{pmatrix} ct' \\ x' \end{pmatrix}$$
which preserve the interval $(ct)^2 - x^2$ are given either by the matrix M of the form
$$M = \Lambda = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$
where $a^2 - b^2 = 1$, or by the matrix M which is the product of Λ and the matrix
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
3. Consider two boosts in the same direction with the rapidity parameters θ_1 and θ_2. Show that they are equivalent to a single boost with the rapidity parameter $\theta = \theta_1 + \theta_2$. What is the rapidity parameter of the transformation inverse to the one with rapidity θ ?
4. Let an event have coordinates $x = 3$km, $t = 1\mu$s in a frame S. What are the coordinates of the same event in a reference frame S' that is moving with the velocity $\vec{v} = (V, 0, 0)$ relative to S ? In which frame does this event happen at $t' = 0$?
5. An astronaut on board of a spaceship of length $L = 300$m has synchronized the rear and front lights so that they flash simultaneously. The spaceship moves with respect to a station with the velocity $v = 10^{-3}c$. When the spaceship passes near the station the lights flash. Would an observer on the station agree that the lights are synchronized ? How big is the desynchronization from his point of view ?
6. Two lights separated by the distance of 1km flash with the difference of 1μs. How fast is an observer moving who thinks that the flashes were emitted simultaneously ?
7. Let the inertial frame S' move with velocity v relative to another inertial frame S. Let a point particle be measured at the space-time points (ct, x) and $(c(t + dt), x + dx)$ in S. Compute these events in the system S'. Assume that the time dt' that has elapsed in the system S' is equal to the time dt in S. Deduce that the velocity u of the particle in S is constant and compute its value. What is the value u' of the velocity in S' ? As the velocity of the particle is constant, the rest frame of the particle is also an inertial frame. Make a Lorentz transformation to that frame and compute the velocities of the two frames S and S' there. Use this to explain the equality of clocks $dt = dt'$.
8. μ-mesons are unstable elementary particles whose lifetime is approximately $\tau \simeq 2 \cdot 10^{-6}$s. They are created in the atmosphere at the height of ~ 10km by cosmic rays. Which velocity should μ-mesons have in order to reach the ground before they decay ?

9. A docking station for space ships consists of a cylinder of the length $L = 100\text{m}$ with super-fast gates at both ends. The gates are synchronized to close and open simultaneously. A space ship of the length $l = 120\text{m}$ is flying through the station at a speed v close to the speed of light. How fast does the ship have to move for the gates to close and lock the ship inside the station (for a brief moment of time)? Consider the same events from the point of view of the ship crew.

10. The space ship crew from the previous problem decides to stop at the station to buy cigarettes. They start breaking when, according to the station observer, the ship is completely inside the station and the gates are closed. They manage to stop the ship before they reach the exit gates. But now the ship does not move and its length is larger than the station. How does it fit inside then?

11. Let the frame S' move with velocity $\vec{V}_1 = (0, V, 0)$ with respect to S, while the frame S'' moves with velocity $\vec{V}_2 = (V, 0, 0)$ with respect to S'. Let the particle velocity in the frame S'' be $\vec{v}'' = (0, v, 0)$. Find the velocity of the particle in the frame S.

12. Let the frames S and S' be related by the boost with the velocity V in the x-direction. In the frame S' a particle is moving with the velocity v' which forms the angle θ' with the x-axis. In the frame S the angle between the particle velocity and the x-axis is θ. Express θ in terms of the velocity v', the angle θ' and the velocity V between the frames.

13. In the previous exercise replace the particle by the light ray. Find the change in the observation angle $\Delta\theta = \theta' - \theta$ (the aberration angle) in the limit $V/c \ll 1$.

14. Consider two inertial frames S and S' with relative velocity v and such that the origins coincide for $t = t' = 0$. After a time Δt an observer at the origin of S sends out a light signal which is received at time $\Delta t'$ by an observer at the origin of S'.

    (a) Suppose first that Galilean relativity holds, i.e. the two frames are related by a Galilean transformation. Compute $\Delta t'$ as a function of Δt and v. (This is the non-relativistic Doppler shift for a receding observer.)

    (b) Deduce now the special relativistic Doppler shift by using Lorentz transformations. The result should be

    $$\Delta t' = \Delta t \sqrt{\frac{1 + v/c}{1 - v/c}}.$$

    Draw a space-time diagram of the situation.