ELECTRODYNAMICS

2009-2010

Literature

1. C. Schomblond "Electrodynamique Classique"
   http:// homepages.ulb.ac.be/~cschomb/notes.html
2. Landau, Lifshitz, vol. II Field Theory
3. Jackson, Classical Electrodynamics

For special relativity see as well

Feynmann Lectures in Physics vol II Space.Time.Motion

I. SPECIAL RELATIVITY

1. Lorentz transformations
2. Formalism: 4-vectors
3. Relativistic particle
4. Particle decay; particle collisions

(1) Lorentz transformations

In order to define <u>motion</u> of a material body, one has to specify a <u>reference frame</u>.

From mechanics we know that there exist a special class of frames called <u>inertial frames</u> which move with respect to one another with constant velocity along sraight lines.

In any of these frames, material bodies move with constant velocities if there are no forces acting on them.

All inertial frames are <u>equivalent</u> to one another. Laws of mechanics look the same in all of them.

This fact may be formulated as the (Galilean) <u>relativity principle</u> : Relative motions of bodies confined in some space do not depend on the motion of this space as a whole if the latter has constant velocity.

$\Rightarrow$ Mechanical laws are the same in all inertial frames.

This can be checked explicitely. The laws of mechanics are summarized in the Newton's law

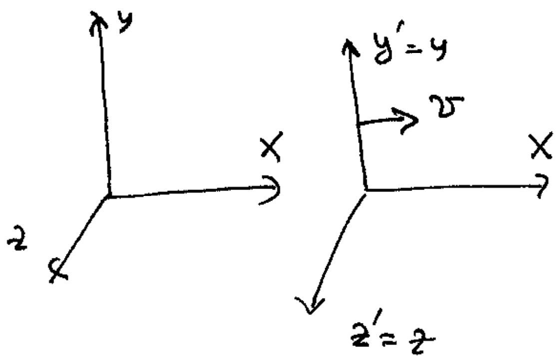
position

$$m \frac{d^2\vec{x}}{dt^2} = \vec{F}$$

mass

force actin on the body.

Position x here is measured with respect to certain <u>reference frame</u>.



$$\boxed{\vec{x} = \vec{x}' + \vec{v}t} \qquad (*)$$

x' – position in a <u>different</u> frame, the one moving with velocity $\vec{v}$.

Galilean transformation

We may calculate the Newton's law in a new (moving) frame.

$$\vec{x} = \vec{x}' + \vec{v}t$$

$$\frac{d\vec{x}}{dt} = \frac{d\vec{x}'}{dt} + \vec{v}$$

$$\frac{d^2\vec{x}}{dt^2} = \frac{d^2\vec{x}'}{dt^2} \Rightarrow$$ Newton's law does not change.

$$\boxed{m\frac{d^2\vec{x}'}{dt^2} = \vec{F}}$$

|| One says that the laws of Newton's mechanics are invariant under the Galilean transformations (2*).

In other words, Newton's mechanics is compatible with the Galilean relativity principle.

A consequence of the transformation (2*) – the velocity addition law:

$$\frac{d\vec{x}}{dt} = \frac{d\vec{x}'}{dt} + \vec{v}$$

So, everything was fine until Maxwell wrote his equations.

It followed from Maxwell's equations that propagation of gravitational waves is described by the equation of the type

$$\left\{ \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right\} E_i = 0 \qquad (*)$$

This equation is <u>NOT</u> compatible with the Galilean relativity principle.

▶ Problem: Show that under the coordinate transformations (2*) this equation changes its form.

⟹ Alternatives:

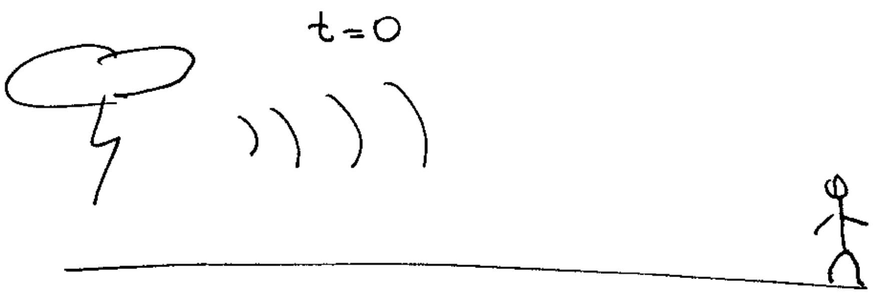
- Maxwell's equations are wrong ?
- relativity principle is wrong ?
- Newtonian mechanics & Galilean transformations are wrong ?

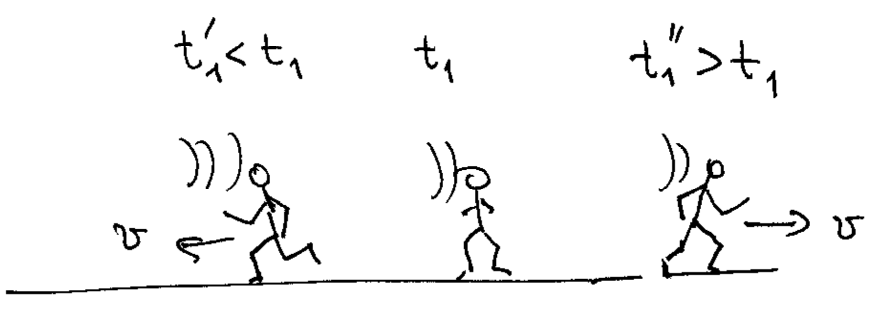
A natural question is: The propagation of sound is described by the equation very similar to (4*)

$$\left\{ \frac{1}{v_s^2} \frac{d^2}{dt^2} - \frac{d^2}{dx^{i\,2}} \right\} p = 0$$

why this does not pose similar problem?

The point is that sound propagates in the <u>air</u>.
The motion with respect to the air is <u>absolute</u>,
$v = 0$ and $v \neq 0$ are two <u>different</u> states.
They may be distinguished by the experiment:

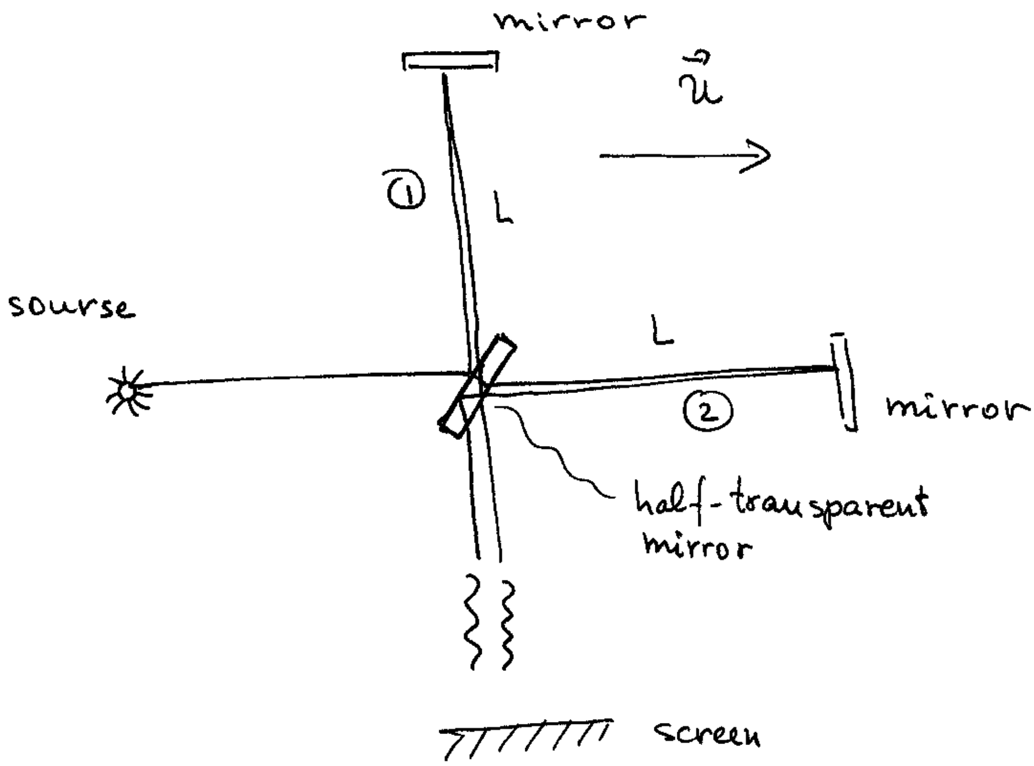




⇒ Relativity is saved if the air is included (if the air moves with us we cannot detect the overall motion).

Can Maxwell equations be interpreted in the same way? This is a matter of experiment.

Michelson & Morley experiment (1881)



If the two ways ① and ② are completely equivalent, then times along two paths are equal and two signals arive in phase.

In the case of motion with respect to "aether" the Galilean law of velocity addition predicts different times.

Zero time shift is observed.

⇒ $\boxed{\text{The principle of relativity works!}}$

⇒ Either Maxwell eqs, or Newtonian dynamics is wrong.

The success of Maxwell theory and failure of the attempts to modify it lead Einstein to the formulation of two postulates:

1. Physics laws are the same in all inertial frames
2. Speed of light is the same and does not depend on the velocity of the frame or of the source.

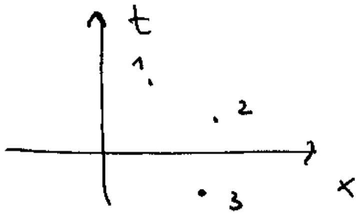
It follows from these postulates that there are the laws of Newtonian mechanics that have to be changed.

The Galilean transformations are <u>not</u> the ones under which the laws of Nature are invariant.

So, what are the correct transformations?

Interval

It is useful to think in terms of "events" - something that happens in a given point of space and in a given time. Events are represented by points on the space-time plot.



Since event has no extent in space and time, it looks the same in all reference frames. For instance, if two particles collide, this happens in all reference frames.

For any two events 1 and 2 one can define an interval S_{12} by the relation

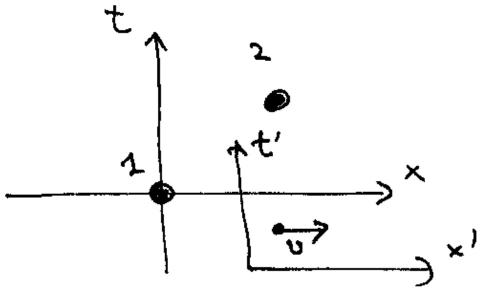
$$S_{12}^2 = c^2 \cdot (t_1 - t_2)^2 - \underbrace{(\vec{x}_1 - \vec{x}_2)^2}$$

distance between two events.

Suppose that:

event 1 = emission of light at the point $(0, \vec{0})$.

event 2 = absorption of light at $(t, \vec{x})$.



Since the speed of light = c we have

$$s_{21}^2 = c^2 \cdot (t-0)^2 - (\vec{x}-\vec{0})^2 = c^2t^2 - \bar{x}^2 = 0$$

Consider <u>the same</u> 2 events from the reference frame t', x' moving with velocity $\vec{v}$. Assume the origins of two systems coincide at $t=0$.

In the moving system:

1 : $0, \vec{0}$

2 : $t', \vec{x}'$

Constant speed of light $\Rightarrow$

$$c^2t'^2 - \vec{x}'^2 = 0.$$

Thus, relation between t', x' and t, x is such that

$$c^2t^2 - \bar{x}^2 = 0 \quad \Rightarrow \quad c^2t'^2 - \bar{x}'^2 = 0.$$

This implies that the interval s_{21}^2 does not depend on the reference frame. Indeed:

1. From homogeneity & isotropy of space-time it follows that $t, \vec{x}$ and $t', \vec{x}'$ are related by a linear transformation of the type

$$t' = \alpha_{11} t + \alpha_{12} x$$
$$x' = \alpha_{21} t + \alpha_{22} x$$

2. $\Rightarrow \quad c^2 t'^2 - \bar{x}'^2 =$ quadratic form of $ct, |\bar{x}|$.

$$= A : (ct + B \cdot |x|)(ct + C \cdot |x|)$$

$A, B, C \rightarrow$ some coefficients.

3. This quadratic form turns zero every time $(ct - |x|)(ct + |x|)$ is zero. Thus,

$$c^2 t'^2 - \bar{x}'^2 = A(|\vec{v}|) \; (c^2 t - \bar{x}^2)$$

$\Rightarrow \quad s'^2 = A(|\bar{v}|) \cdot s^2$. i.e., intervals are proportional with the coefficient depending on $|\vec{v}|$

4. Consider two moving frames with velocities $\vec{v}_1 \nparallel \vec{v}_2$.

Then

$$s'^2 = A(|v_1|)\, s^2$$

$$s''^2 = A(|v_2|)\, s^2$$

and also

$$s'^2 = A(|\bar{v}_1 - \bar{v}_2|)\, s''^2 .$$

$$\Rightarrow \quad A(|\bar{v}_1 - \bar{v}_2|) = \frac{A(|v_1|)}{A(|v_2|)}$$

depends on $\cos(\widehat{v_1 v_2})$

does not depend on $\cos(\widehat{v_1 v_2})$

$$\Rightarrow \quad A(v) = \text{const} = 1.$$

$$\Rightarrow \quad \boxed{s'^2 = s^2}$$

Interval does not depend on the reference frame

From this point of view one may consider interval as a sort of "distance" in space-time.

Lorentz transformations

They follow immediately from the conservation of the interval. Consider the case $\vec{v} = (v,0,0)$ (if this is not the case originally, we can turn the space coordinates). Then

$$(*) \quad \begin{aligned} x &= x' \cdot \mathrm{ch}\,\theta + ct'\,\mathrm{sh}\,\theta \\ ct &= x'\,\mathrm{sh}\,\theta + ct'\,\mathrm{ch}\,\theta \end{aligned}$$

"rapidity"

$$\begin{pmatrix} \mathrm{ch}\,\theta = \frac{1}{2}(e^{\theta} + e^{-\theta}) \\ \mathrm{sh}\,\theta = \frac{1}{2}(e^{\theta} - e^{-\theta}) \end{pmatrix}$$

$$\mathrm{ch}^2\theta - \mathrm{sh}^2\theta = 1.$$

or, in the matrix form

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \mathrm{ch}\,\theta & \mathrm{sh}\,\theta \\ \mathrm{sh}\,\theta & \mathrm{ch}\,\theta \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}$$

cf: rotation!

Clearly, the interval is conserved:

$$c^2t^2 - x^2 = (x'\,\mathrm{sh}\,\theta + ct'\,\mathrm{ch}\,\theta)^2 - (x'\,\mathrm{ch}\,\theta + ct'\,\mathrm{sh}\,\theta)^2 =$$

$$= x'^2\,\mathrm{sh}^2\theta + 2x'ct'\,\mathrm{sh}\,\theta\,\mathrm{ch}\,\theta + c^2t'^2\,\mathrm{ch}^2\theta$$

$$- x'^2\,\mathrm{ch}^2\theta - 2x'ct'\,\mathrm{sh}\,\theta\,\mathrm{ch}\,\theta - c^2t'^2\,\mathrm{sh}^2\theta =$$

$$= c^2t'^2(\mathrm{ch}^2\theta - \mathrm{sh}^2\theta) - x'^2(\mathrm{ch}^2\theta - \mathrm{sh}^2\theta)$$

$$= c^2t'^2 - x'^2 \qquad \text{Ok}$$

Thus, the transformations (12*) preserve the interval. It remains to determine how β is related to the velocity v.

We have a frame $(x't')$ moving with respect to our frame (t,x) with the velocity v. Consider the motion of the <u>origin</u> $(x'=0)$ of the frame (t',x'). Since $x'=0$ we have

$$x = ct' \cdot \operatorname{sh}\theta$$

$$ct = ct' \cdot \operatorname{ch}\theta$$

But x/t is the velocity v of the moving system! Dividing the two equations we get

$$\frac{v}{c} = \frac{\operatorname{sh}\theta}{\operatorname{ch}\theta} = \operatorname{th}\theta$$

Thus we get $\theta = \operatorname{arcth}\frac{v}{c}$. The coefficients of the Lorentz transformation (12*) are $\operatorname{sh}\beta$ and $\operatorname{ch}\beta$. Express them in terms of $v/c \equiv \beta$

$$\operatorname{ch}^2\theta - \operatorname{sh}^2\theta = 1$$

$$1 - \operatorname{th}^2\theta = \frac{1}{\operatorname{ch}^2\theta} = 1 - v^2/c^2$$

$$\operatorname{ch}\theta = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

$$\mathrm{sh}\,\theta = \sqrt{1-\mathrm{ch}^2\theta} = \frac{v/c}{\sqrt{1-v^2/c^2}} = \frac{\beta}{\sqrt{1-\beta^2}}$$

<u>Note</u>: One also uses notations $\beta = v/c$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

Finally, Lorentz transformations are:

$$\boxed{\begin{aligned} x &= \frac{x' + ct'\cdot\frac{v}{c}}{\sqrt{1-v^2/c^2}} \\ ct &= \frac{x'\cdot v/c + ct'}{\sqrt{1-v^2/c^2}} \end{aligned}} \quad \begin{aligned} &= \frac{x' + \beta\cdot ct'}{\sqrt{1-\beta^2}} \\ &= \frac{x'\beta + ct'}{\sqrt{1-v^2/c^2}} \end{aligned}$$

$$\boxed{\begin{aligned} y &= y' \\ z &= z' \end{aligned}}$$

Consequences

1. Light cone

Consider two events $(t_1, \vec{x}_1)$ and $(t_2, \vec{x}_2)$. Is there a reference frame such that $\bar{x}_1' = \bar{x}_2'$? That is, they happened in the same point in space?

We must have

$$c^2(t_2-t_1)^2 - (\bar{x}_2-\bar{x}_1)^2 = c^2(t_2'-t_1')^2 - \underbrace{\cancel{(\bar{x}_2'-\bar{x}_1')^2}}_{=0} > 0$$

Thus, such a frame exists if $S_{21}^2 > 0$. Such intervals are called <u>time-like</u>.

‖ The events happening with the same material body are always separated by time-like intervals.

If the interval between 2 events is negative, then there exists such a frame that $t_2' = t_1'$, so that

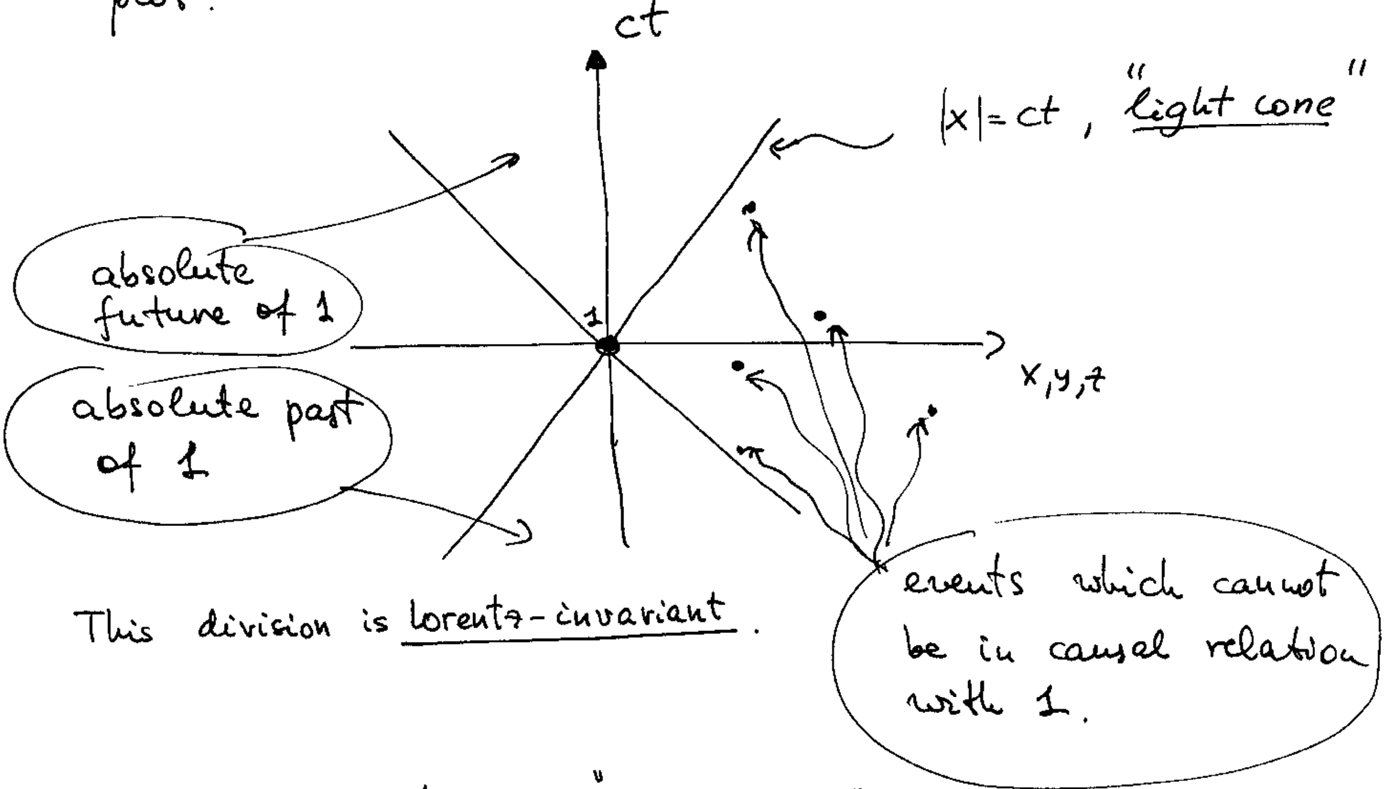
$$c^2(t_2-t_1)^2 - (\bar{x}_2-\bar{x}_1)^2 = -(\bar{x}_2'-\bar{x}_1')^2 < 0$$

Such intervals are called <u>space-like</u>.

⇒ one may have $t_2 > t_1$ and at the same time $t_2' < t_1'$. In other words, "earlier" and "later" are relative notions for such events.

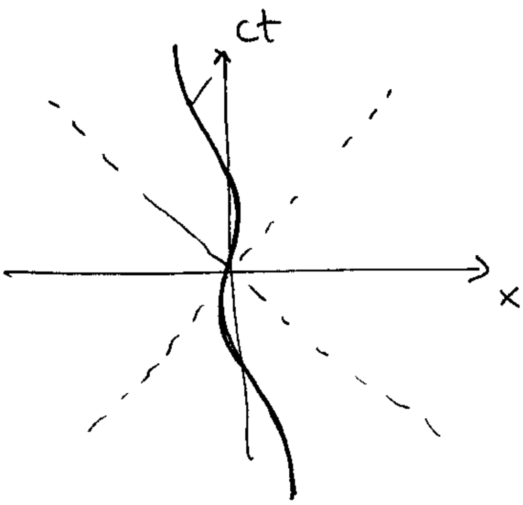
Consider two events separated by a time-like interval. Let $t_2 > t_1$ in some reference frame. Then $t_2' > t_1'$ in all frames (otherwise there must exist a frame in which $t_2' = t_1'$, which is impossible for a time-like interval).

All this can be summarized on the following plot:



This division is Lorentz-invariant.

history of a particle - "world line"



2. Proper time

Consider a particle (body, ...) moving with the time-dependent velocity $\vec{v}(t)$. Two close points on particle world line are separated by the time dt and $d\vec{x} = dt \cdot \vec{v}(t)$; the interval between them is

$$ds = \sqrt{c^2 dt^2 - v^2 dt^2} = c\, dt \sqrt{1 - v^2/c^2} .$$

In the frame which at the time t moves with the velocity $\vec{v}$ the particle is at rest, $d\vec{x}' = 0$. The time which elapses is $c dt' = ds = c dt \sqrt{1 - v^2/c^2} \equiv c d\tau$. It is called proper time τ. For finite intervals one has

$$t_2 - t_1 = \int_{\tau_1}^{\tau_2} \frac{d\tau}{\sqrt{1 - v^2/c^2}}$$

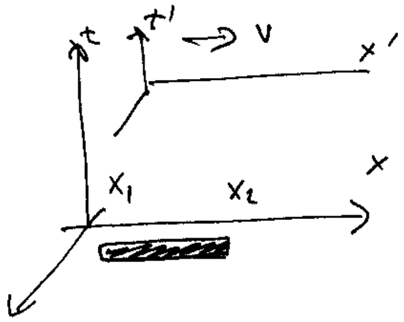
Since $1/\sqrt{1 - v^2/c^2} \geq 1$ we have $t_2 - t_1 \geq \tau_2 - \tau_1$.

$\Rightarrow$ The time passed in the body's internal frame is shorter than in the reference frame where the body moves. This is the Lorentz time dilation.

This effect is observed. For instance, μ-mesons are produced in the atmosphere by cosmic rays at heights of several ($\gtrsim 10$) kilometers. μ has a life time $\tau_\mu = 2 \cdot 10^{-6}\,s$; with this lifetime and velocity of light it can propagate $\sim 600\,m$. How can it reach the ground, i.e. travel $\gtrsim 10$ km? This is possible because the time (between creation and decay) in laboratory frame (where μ is moving) is longer than in the μ-meson proper frame.

3. Lorentz contraction. Proper length

Consider a bar at rest. The bar length is



$$\Delta x = x_2 - x_1 .$$

Let us find the length of the same bar from the point of view of an observer moving with the velocity v. This observer has to take the coordinates of the ends of the bar at the same moment of time t' and subtract them.

we have from Lorentz transformations:

$$x_1 = \frac{x_1' + vt'}{\sqrt{1 - v^2/c^2}}$$

$$x_2 = \frac{x_2' + vt'}{\sqrt{1 - v^2/c^2}}$$

Subtracting one from the other

$$x_2 - x_1 = \Delta x = \frac{x_2' - x_1'}{\sqrt{1 - v^2/c^2}} = \frac{\Delta x'}{\sqrt{1 - v^2/c^2}}$$

$\Rightarrow$ The apparent length of the bar in the system where it moves is <u>shorter</u> than the length in the bar rest frame. This is called <u>Lorentz contraction</u>. The length in the comoving frame is called <u>proper length</u>

4. Velocity composition

Let the frame (t', x') moves with respect to the frame (t, x) with the velocity V in the direction of the x-axis: $\vec{V} = (V, 0, 0)$.

Consider a moving particle. Its velocity is

$$v_x = \frac{dx}{dt} \quad \text{- in the frame } (t, x)$$

$$v_x' = \frac{dx'}{dt'} \quad \text{- in the frame } (t', x').$$

Lorentz transformations give

$$dx = \frac{dx' + V\,dt'}{\sqrt{1 - V^2/c^2}}$$

$$dy = dy'$$

$$dz = dz'$$

$$dt = \frac{dt' + \frac{V}{c^2}\,dx'}{\sqrt{1 - V^2/c^2}}$$

Devide three first equations by the 4-th one:

$$v_x = \frac{dx}{dt} = \frac{dx' + V\,dt'}{dt' + V/c^2\,dx'} = \frac{v_x' + V}{1 + \frac{V v_x'}{c^2}}$$

$$v_y = \frac{dy}{dt} = \frac{dy'}{dt' + V/c^2\,dx'}\sqrt{\dots} = \frac{v_y'}{1 + \frac{V v_x'}{c^2}} \cdot \sqrt{1 - \frac{V^2}{c^2}}$$

$$v_z = \frac{v_z'}{1 + \frac{V v_x'}{c^2}}\sqrt{1 - V^2/c^2}$$

Remarks:

1. Let V be much less than c, $V \ll c$. Then we can expand these relations:

$$v_x \approx (v_x' + V)\left(1 - \frac{V}{c}\cdot\frac{v_x'}{c} + \dots\right) = v_x' + V - \frac{V}{c}\frac{v_x'}{c}(v_x' + V) + \dots$$

$$v_y \approx v_y'\left(1 - \frac{V v_x'}{c^2}\right)\left(1 - \frac{V^2}{2c^2}\right) \approx v_y'\left(1 - \frac{V v_x'}{c^2} - \frac{V^2}{2c^2} + \dots\right)$$

$$v_z \approx v_z'\left(1 - \frac{V v_x'}{c^2} - \frac{V^2}{2c^2} + \dots\right).$$

Thus, when $V \ll c$ we recover the Galilean law

$$v_x = v_x' + V$$

$$v_y = v_y'$$

$$v_z = v_z'$$

2. What happens when the velocity V is close to c, $V \sim c$, $c - V \ll c$? Can v_x become larger than c ?

Take
$$v_x' = (1-\varepsilon)c \qquad V = (1-\varepsilon)c$$
where $\varepsilon \ll 1$.

we have

$$v_x = \frac{2(1-\varepsilon)c}{1 + (1-\varepsilon)^2} = \frac{2(1-\varepsilon)c}{1 + 1 - 2\varepsilon + \varepsilon^2} =$$

$$= \frac{2(1-\varepsilon)}{2(1-\varepsilon) + \varepsilon^2} \cdot c$$

This is of order, but still smaller than c.

$\Rightarrow$ The velocity which is smaller than c in one frame, is smaller than c in all frames.

3. Finally, consider the case when $v_x' = c$. Then

$$v_x = \frac{c + V}{1 + \frac{Vc}{c^2}} = c\,\frac{1 + V/c}{1 + V/c} = c$$

Thus indeed the light velocity is constant.

Formalism of 4-vectors

We will proceed by analogy with space rotations, so let us recall the logic in that case.

Points in space are associated with 3-vectors

$$\vec{x} = (x^1, x^2, x^3) = (x, y, z)$$

The length of a vector (distance from the origin) is defined as $|\vec{x}|$,

$$|\vec{x}|^2 = \sum_i x^i x^i = \sum_{i,j} \delta_{ij}\, x^i x^j \qquad (*)$$

where

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{unit matrix } 3\times 3$$

Rotations are linear transformations that preserve the distance $(*)$:

$$x^i = \sum_j R^i{}_j \cdot x^j \qquad \Bigg| \quad \text{matrix form:} \quad x = R x' \qquad (**)$$

Here $R^i{}_j = R^i{}_j(\psi, \theta, \varphi)$ is a 3×3 matrix of rotation depending on 3 rotation angles ψ, θ, φ. Invariance of the distance requires that R obeys

$$R^T R = 1$$

Indeed,

$$\sum_i x^i x^i = \sum_{i,j,k} R^i{}_j\, x'^j\, R^i{}_k\, x'^k =$$

$$= \sum_{i,j,k} R^i{}_j R^i{}_k\, x'^j x'^k =$$

$$= \sum_{j,k} \underbrace{\left(\sum_i R^i{}_j R^i{}_k \right)}_{\text{must be equal to } \delta_{jk} \text{ for the result to be equal to } x'^j x'^j} x'^j x'^k$$

$$\sum_i R^i{}_j R^i{}_k = \delta_{jk} \quad \rightarrow \quad R^T R = 1 \quad \text{(matrix form)}.$$

Remark: a widely accepted convention is that one sums over two repeating indices (unless stated otherwise). That is

$$x^i x^i \equiv \sum_i x^i x^i$$

$$R^i{}_j\, x^j\, R^i{}_k\, x^k \equiv \sum_{i,j,k} R^i{}_j\, x^j\, R^i{}_k\, x^k$$

....

Scalar product of two vectors is

$$\vec{x}\vec{y} = x^i y^i$$

It is invariant under rotations.

<u>key point</u>: vectors are quantities that transform like (24**) under rotations.

Space-time and Lorentz symmetry

* Points in space-time correspond to contravariant 4-vectors

$$x^\mu = (ct, x^1, x^2, x^3) \equiv (x^0, x^1, x^2, x^3)$$

* Invariant quantity (analog of the distance) is interval

$$s^2 = (ct)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

It can be written as

$$s^2 = \sum_{\mu,\nu} \eta_{\mu\nu} x^\mu x^\nu = \eta_{\mu\nu} x^\mu x^\nu$$

where $\mu, \nu = 0, 1, 2, 3$ and

$$\eta_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The matrix $\eta_{\mu\nu}$ is called metric tensor of Minkowski space.

* One defines also a covariant vector

$$x_\mu = \eta_{\mu\nu} x^\nu = (ct, -x^1, -x^2, -x^3)$$

Making use of covariant vector the interval may be written simply as

$$S^2 = x_\mu x^\mu$$

* The scalar product of two 4-vectors x^μ and y^μ is defined as

$$(x \cdot y) \equiv \eta_{\mu\nu} x^\mu y^\nu = x_\mu y^\mu = x^\mu y_\mu$$

* One defines $\eta^{\mu\nu}$ as the matrix inverse of $\eta_{\mu\nu}$, so that

$$\eta^{\mu\lambda} \eta_{\lambda\nu} = \delta^\mu_{\ \nu} \hookrightarrow \text{unit matrix } 4\times4$$

Clearly,

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Making use of $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ one may transform covariant vectors to contravariant and vise versa:

$$x^\mu = \eta^{\mu\nu} x_\nu$$

$$x_\mu = \eta_{\mu\nu} x^\nu$$

* Lorentz transformations are defined as linear transformations preserving the interval.

One has

$$x^\mu = \underbrace{\Lambda^\mu{}_\nu}_{4\times 4 \text{ matrix}} \cdot x'^\nu$$

matrix form:

$$x = \Lambda x' \qquad (*)$$

For instance, boost in the direction x is written as

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ch\theta & sh\theta & 0 & 0 \\ sh\theta & ch\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

* Defining property of matrices $\Lambda^\mu{}_\nu$: from invariance of interval one has:

$$\eta_{\mu\nu} x^\mu x^\nu = \eta_{\mu\nu} x'^\mu x'^\nu$$

Substituting $(*)$ to the left hand side:

$$\eta_{\mu\nu} \underbrace{\Lambda^\mu{}_\lambda x'^\lambda}_{x^\mu} \cdot \underbrace{\Lambda^\nu{}_\rho x'^\rho}_{x^\nu} = \eta_{\mu\nu} x'^\mu x'^\nu$$

$$\Rightarrow \quad \boxed{\begin{array}{l} \eta_{\mu\nu} \Lambda^\mu{}_\lambda \Lambda^\nu{}_\rho = \eta_{\lambda\rho} \\ \text{matrix form: } \quad \Lambda^T \eta \Lambda = \eta \end{array}}$$

Consequences: invariance of the scalar product

$$(x \cdot y) = \eta_{\mu\nu} x^\mu y^\nu = \eta_{\mu\nu} \Lambda^\mu{}_\lambda \Lambda^\nu{}_\rho x'^\lambda y'^\rho =$$

$$= \eta_{\lambda\rho} x'^\lambda x'^\rho = (x' \cdot y')$$

Transformation law for covariant vectors:

$$(x \cdot y) = x_\mu y^\mu = \text{invariant} = x'_\mu y'^\mu$$

$$x_\mu y^\mu = \underbrace{x_\mu \Lambda^\mu{}_\nu}_{x'_\mu} y'^\nu$$

$$x'_\mu = x_\mu \Lambda^\mu{}_\nu$$

(*) $$x_\mu = x'_\nu (\Lambda^{-1})^\nu{}_\mu$$

matrix form: $x^T = x'^T \cdot \Lambda^{-1}$

* **Tensors** are objects whose components transform like products of vectors. For instance, a tensor $t^{\mu\nu}$ with two contravariant indices transforms like $x^\mu y^\nu$, so that

$$t^{\mu\nu} = \Lambda^\mu{}_\lambda \cdot \Lambda^\nu{}_\rho \cdot (t')^{\lambda\rho}$$

* Tensors $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ are <u>invariant</u> under Lorentz transformations. Indeed,

$$(x \cdot y) = \eta_{\mu\nu} x^\mu y^\nu = \underbrace{\eta_{\mu\nu} \Lambda^\mu{}_\lambda \Lambda^\nu{}_\rho}_{} x'^\lambda x'^\rho$$

by invariance of scalar product

$= \eta'_{\lambda\rho}$ by the law of transformation of covariant components.

$$= (x' \cdot y') = \eta_{\mu\nu} x'^\mu y'^\nu$$

$$\Rightarrow \quad \eta'_{\mu\nu} = \eta_{\mu\nu}$$

* Another invariant tensor is $\epsilon^{\mu\nu\lambda\rho}$ defined as follows

1) $\epsilon^{\mu\nu\lambda\rho}$ is absolutely antisymmetric, that is $\epsilon^{\mu\nu\lambda\rho} = -\epsilon^{\nu\mu\lambda\rho}$ etc.

2) $\epsilon^{0123} = 1$.

$$\Rightarrow \epsilon^{\mu\nu\lambda\rho} = \begin{cases} 0 & \text{if any two indices coincide} \\ 1 & \text{if } \mu\nu\lambda\rho \text{ is even permutation of } 0123 \\ -1 & \text{if } \mu\nu\lambda\rho \text{ is odd permutation of } 0123 \end{cases}$$

* <u>Invariance of $\epsilon^{\mu\nu\lambda\rho}$</u>. Consider the result of transformation of $\epsilon^{\mu\nu\lambda\rho}$,

$$A^{\alpha\beta\gamma\delta} = \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} \Lambda^{\gamma}{}_{\lambda} \Lambda^{\delta}{}_{\rho} \epsilon^{\mu\nu\lambda\rho}$$

- Tensor $A^{\alpha\beta\gamma\delta}$ is absolutely antisymmetric

$$A^{\alpha\beta\gamma\delta} = -A^{\beta\alpha\gamma\delta}, \ \ldots \text{ etc.}$$

$$\Rightarrow \quad A^{\alpha\beta\gamma\delta} = \text{Const} \cdot \epsilon^{\alpha\beta\gamma\delta}$$

- The component

$$A^{0123} = \underbrace{\Lambda^{0}{}_{\mu} \Lambda^{1}{}_{\nu} \Lambda^{2}{}_{\lambda} \Lambda^{3}{}_{\rho} \epsilon^{\mu\nu\lambda\rho}}_{\text{definition of } \det(\Lambda)\ !} = \det\Lambda$$

$$\Rightarrow \quad \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} \Lambda^{\gamma}{}_{\lambda} \Lambda^{\delta}{}_{\rho} \epsilon^{\mu\nu\lambda\rho} = \det\Lambda \cdot \epsilon^{\alpha\beta\gamma\delta}$$

Now, consider the relation

$$\Lambda^{T} \eta \Lambda = \eta$$

$$\Rightarrow \quad \det\Lambda \cdot \det\eta \cdot \det\Lambda = \det\eta$$

$$\Rightarrow \quad \det\Lambda = \pm 1.$$

$$\boxed{\Rightarrow \epsilon^{\mu\nu\lambda\rho} \text{ is invariant under transformations that have } \det\Lambda = 1}$$

* One also defines $\epsilon_{\mu\nu\lambda\rho} = \eta_{\mu\alpha}\,\eta_{\nu\beta}\,\eta_{\lambda\gamma}\,\eta_{\rho\delta}\,\epsilon^{\alpha\beta\gamma\delta}$.

This tensor is also absolutely antisymmetric and has

$$\epsilon_{0123} = -1 \qquad \text{(explain why!)}$$

* An important example of covariant vector is the <u>gradient</u> $\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$. To see that this vector is covariant, act by ∂_μ on a scalar (e.g., scalar product $x \cdot y$)

$$\partial_\mu (x \cdot y) = \frac{\partial}{\partial x^\mu}\, x^\nu \cdot y_\nu = \delta_\mu{}^\nu\, y_\nu = y_\mu$$

<u>To summarize</u> :

- contravariant vectors transform under Lorentz transformation as x^μ
- covariant vectors transform as given in (28*)
- contraction of covariant and contravariant index produces a scalar
- tensors with several indices transform like product of vectors with the same indices.
- <u>All terms in a Lorentz-invariant equation must carry the same number of covariant and contravariant indices</u>.

RELATIVISTIC MECHANICS

I). Velocity

Let us first generalize the velocity of a particle to the relativistic case. The velocity itself does not have simple Lorentz transformation properties:

$$\vec{v} = \frac{dx^i}{dt}$$

This is seen already from the velocity composition law, which is not similar to the transformation of space components of a 4-vector like, for instance, x^i.

One can compose the quantity which is a 4-vector and whose space part becomes a velocity in the non-relativistic limit.

dx^μ — 4-vector

ds — scalar

$$\Downarrow$$

$u^\mu \equiv c\,\dfrac{dx^\mu}{ds}$ — 4-vector. (i.e., transforms by the same matrix $\Lambda^\mu{}_\nu$ as x^μ).

This is called 4-velocity

Explicitely,

$$u^\mu = c\,\frac{(c\,dt,\ dx^i)}{ds} =$$

$$\left(ds = c\,dt\sqrt{1-\left(\frac{dx^i}{c\,dt}\right)^2} = c\,dt\sqrt{1-\frac{v^2}{c^2}} \right.$$

$$= \left(\frac{c}{\sqrt{1-v^2/c^2}} \ ; \ \frac{v^i}{\sqrt{1-v^2/c^2}} \right)$$

when $v^2 \ll c^2$ we have

$$u^\mu \simeq (c, v^i) \quad \to \text{Ok}.$$

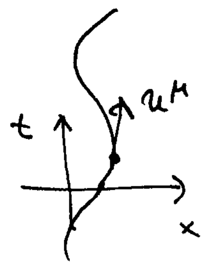
The components of u^μ satisfy the property

$$(u\cdot u) = u_\mu u^\mu = c^2$$

Indeed,

$$u_\mu u^\mu = c^2 \frac{dx_\mu}{ds}\cdot\frac{dx^\mu}{ds} = c^2\frac{dx_\mu dx^\mu}{ds^2} = c^2\frac{ds^2}{ds^2} = c^2$$

In geometrical terms, the velocity u^μ is a tangent vector to the particle world line.



2. Energy & momentum

In non-relativistic mechanics the momentum is related to velocity by

$$p^i = m v^i$$

One may expect similar relation to hold in the relativistic case as well,

$$p^\mu = m u^\mu .$$

Let us demonstrate that this is indeed the case. To be more precise, let us show that this is the only form compatible with Lorentz invariance and energy-momentum conservation, and having correct non-relativistic limit.

Because of rotational invariance (absence of a preferred direction) we may write

$$p^i = \mathcal{M}(v) \cdot v^i$$

$$E = \mathcal{E}(v)$$

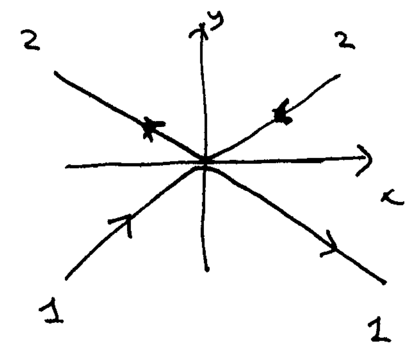
where the functions $\mathcal{M}(v)$ and $\mathcal{E}(v)$ have to be determined.

At small velocities we must have

$$\mathcal{M}(v) = m + \dots$$

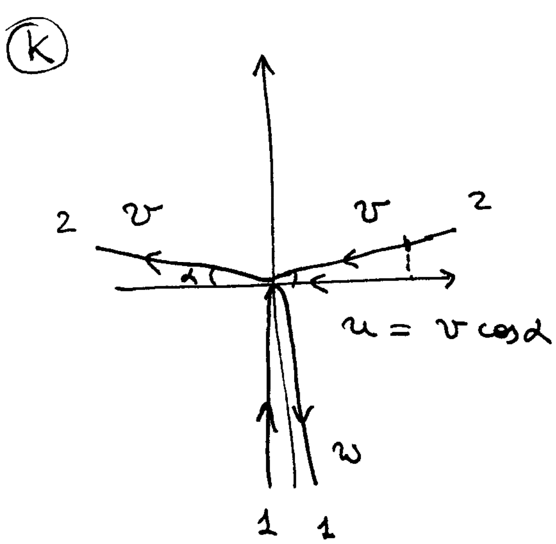
$$\mathcal{E}(v) = \text{const} + \frac{1}{2} v^2 \cdot m + \dots$$

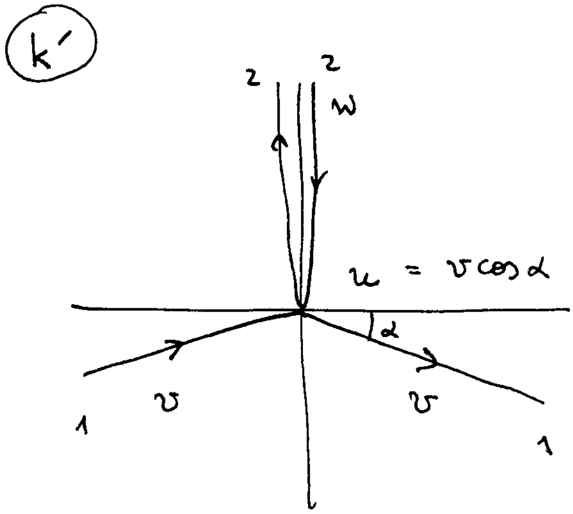
In order to find the dependence of $\mathcal{M}$ and $\mathcal{E}(v)$ on $v = |\vec{v}|$ consider collision of two identical particles. In the center-of-mass frame this collision looks as follows:



Because masses of particles are the same, all 4 velocities have equal modulus.

We may look at this collision in the frame k where "horizontal" component of the particle 1 is zero – this will simplify calculations.





In the frame k the conservation of momentum gives:

$$x: \quad 0 + \mathcal{M}(v)\cdot v\cos\alpha = \mathcal{M}(v)\, v\cos\alpha + 0$$

$$y: \quad \mathcal{M}(w)w - \mathcal{M}(v)\, v\sin\alpha = \mathcal{M}(v)\, v\sin\alpha - \mathcal{M}(w)\, w$$

The last equation gives:

$$\mathcal{M}(w)\cdot w = \mathcal{M}(v)\cdot \underbrace{v\sin\alpha}$$

(we need to find this in terms of w

In order to find $v\cdot\sin\alpha$ consider another frame K' where now particle 2 has zero v_x'.

From velocity composition law we have for particle 2:

$v_x' = 0!$

$$v\cos\alpha = v_x = \frac{v_x' + V}{1 + \frac{v_x' V}{c^2}} = V \quad \Rightarrow \quad V = v_x = v\cos\alpha$$

$$v\sin\alpha = v_y = \frac{v_y'}{1 + \frac{v_x' V}{c^2}}\sqrt{1 - \frac{V^2}{c^2}} = w\cdot\sqrt{1 - \cos^2\alpha\,\frac{v^2}{c^2}}$$

Thus:

$$\mathcal{M}(w)\cdot w = \mathcal{M}(v)\, w\cdot\sqrt{1 - \cos^2\alpha\,\frac{v^2}{c^2}}$$

Now remember that the collision can be arranged in such a way that the angle α takes a given value. Therefore, we may consider the limit $\alpha \to 0$. In this limit

$$w = v \cdot \sin\alpha \simeq v \cdot \alpha + \ldots$$

$$\cos^2\alpha = 1 + \ldots$$

Thus we find

$$\underbrace{\mathcal{M}(v \cdot \alpha)}_{} \cdot \cancel{v} \cdot \cancel{\alpha} = \mathcal{M}(v) \cdot \cancel{v} \cdot \cancel{\alpha} \sqrt{1 - \frac{v^2}{c^2}} + \ldots$$

$\hookrightarrow$ goes to 0 as $\alpha \to 0 \Rightarrow \mathcal{M}(0) = m$

$$\Rightarrow \boxed{\mathcal{M}(v) = \frac{m}{\sqrt{1 - v^2/c^2}}}$$

Thus, the relativistic momentum is

$$\boxed{p^i = \frac{m v^i}{\sqrt{1 - v^2/c^2}}}$$

<u>Note</u>: p^i grows infinitely when v approaches c. Thus, while v is always less than c, p^i can be <u>infinitely large</u>.

Now we have to determine the relativistic expression for the energy of a moving object. One can do calculations similar (but more involved) to those we have done for the momentum, but this is not necessary. We just have to find a conserved quantity which goes to the nonrelativistic expression for the energy in the limit of small velocity.

First we note that

$$P^i = m u^i$$

where u^i are the space components of the 4-velocity.

What is the meaning of $P^0 = m u^0$? Recall that

$$u^\mu = \left(\frac{c}{\sqrt{1-v^2/c^2}} \, ; \, \frac{v^i}{\sqrt{1-v^2/c^2}} \right)$$

so that

$$P^0 = \frac{mc}{\sqrt{1-v^2/c^2}} = m u^0$$

Let us expand this expression at small velocity $v \ll c$:

$$P^0 \simeq cm \left(1 - v^2/c^2\right)^{-1/2} \simeq cm \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots\right)$$

$$= mc + \underbrace{\frac{1}{2} m v^2}_{\text{non-relativistic energy!}} \cdot \frac{1}{c} + \ldots$$

Let us see that P^0 is conserved. Consider some process (e.g., collision) in which initial and final momenta are P^i_{in} and P^i_{fin}. Let us make a boost in the x-direction and use the fact that momentum is conserved in both frames. We have

$$P^x_{in} = P^x_{fin} \qquad \text{(momentum conservation in the original frame)}$$

$$\frac{P^x_{in} + \frac{V}{c} P^0_{in}}{\sqrt{1 - V^2/c^2}} = \frac{P^x_{fin} + \frac{V}{c} P^0_{fin}}{\sqrt{1 - V^2/c^2}} \qquad \text{(momentum conservation after boost)}$$

$$\Rightarrow \quad P^0_{in} = P^0_{fin}$$

Therefore, P^0 must be the energy up to a normalization. Comparing to the low-velocity expansion we find

$$c \cdot P^0 = \mathcal{E}(v) = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = E$$

Thus, we have

$$(*) \qquad P^\mu = \left(\frac{E}{c} , P^i \right) = \left(\frac{mc}{\sqrt{1 - v^2/c^2}} ; \frac{m v^i}{\sqrt{1 - v^2/c^2}} \right).$$

The expression for the relativistic energy can be derived in yet another way, starting from the relation between the change of momentum and the change of energy. Energy change equals the performed work,

$$dE = \vec{F}\cdot d\vec{x} = \frac{d\vec{P}}{dt}\cdot d\vec{x} = d\vec{P}\cdot\frac{d\vec{x}}{dt} = \vec{v}\cdot d\vec{P}$$

Now let us use the relativistic expression for the momentum,

$$dE = \vec{v}\, d\left(\frac{m\vec{v}}{\sqrt{1-v^2/c^2}}\right) =$$

$$= m\vec{v}\left\{\frac{d\vec{v}}{\sqrt{1-v^2/c^2}} + \frac{v^2/c^2\, d\vec{v}}{(1-v^2/c^2)^{3/2}}\right\} =$$

$$= m\vec{v}\ \frac{1-\cancel{v^2/c^2}+\cancel{v^2/c^2}}{(1-v^2/c^2)^{3/2}}\, d\vec{v} =$$

$$= m\frac{\vec{v}\,d\vec{v}}{(1-v^2/c^2)^{3/2}} = d\left(\frac{mc^2}{\sqrt{1-v^2/c^2}}\right)$$

Integrating, we have

$$E = \frac{mc^2}{\sqrt{1-v^2/c^2}},$$

the same expression as before.

Remarks

1. Energy is <u>not</u> a Lorentz invariant, it transforms as a 0-th component of a vector. Since $\left(\frac{E}{c}, p^i\right)$ form a contravariant vector p^μ we can write the transformation of energy-momentum from one frame to the other as follows:

$$p^0 = \frac{E}{c} = \frac{\tilde{E}/c + \frac{V}{c}\tilde{p}^1}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$p^1 = \frac{\tilde{p}^1 + \frac{V}{c}\cdot\frac{\tilde{E}}{c}}{\sqrt{1 - V^2/c^2}}$$

2. From the expression

$$E = \frac{mc^2}{\sqrt{1 - V^2/c^2}}$$

we see that at $v \to 0$ energy is not zero but goes to a finite constant

$$E_0 = mc^2$$

This energy is called "rest enenergy", or "rest mass".

The energy stored in the form of mass is as real as the kinetic energy as it can be transformed to the latter, at least in principle. Imagine two light bodies moving toward each other with a large velocity v. The energy in the center of mass is $2 \times \frac{mc^2}{\sqrt{1-v^2/c^2}} \gg 2mc^2$ – i.e, this energy is mostly "kinetic". If after collision the bodies stay together (for instance, spin around each other), the energy conservation implies $E_{final} = \frac{2mc^2}{\sqrt{1-v^2/c^2}} = Mc^2$, since the final velocity is zero. Thus, the kinetic energy got converted to the "mass". The inverse process is equally possible (for instance, nuclear decay).

3. From the explicit expression (39*) we find that

$$(P \cdot P) = P_\mu P^\mu = P^{o2} - P^{i2} =$$

$$= \frac{m^2c^2}{1-v^2/c^2} - \frac{m^2 v_i^2}{1-v^2/c^2} = m^2c^2 \frac{1-v^2/c^2}{1-v^2/c^2} = m^2c^2$$

is Lorentz-invariant (since it is the square of a 4-vector). Thus we get the following relation:

$$m^2c^2 = \frac{E^2}{c^2} - \vec{P}^2$$

or

$$\boxed{m^2 c^4 = E^2 - c^2 \vec{p}^{\,2}}$$ – the relativistic relation between energy and momentum.

Another useful relation:

$$\vec{p} = \frac{E \cdot \vec{v}}{c^2} \qquad (*)$$

4. The energy associated with the mass of a macroscopic body is enormous.

$$1g \cdot c^2 = 10^{-3} \cdot (3 \cdot 10^8)^2 \cdot J \simeq 10^{-3} \cdot 10^{17} J = 10^{14} J$$

For comparison: burning of 1 kg of wood produces $\sim 10^7$ J.

$\Rightarrow$ 1g $\Leftrightarrow$ 10^4 tonn of wood!

Relativistic version of the Newton's law

Now it is not difficult to write down the relativistic analog of the Newton's law. Such a generalization has to be written in terms of 4-vectors. It reads

$$mc\,\frac{du^\mu}{ds} = f^\mu \qquad (*)$$

The 4 components of the force f^μ are not all independent. Indeed, upon multiplication by u_μ eq. (*) gives

$$mc\,\underbrace{u_\mu \frac{du^\mu}{ds}}_{=0} = u_\mu f^\mu$$

$=0$ because $u_\mu \dfrac{du^\mu}{ds} = \dfrac{1}{2}\dfrac{d}{ds}\underbrace{(u_\mu u^\mu)}_{=1} = 0.$

Thus, f^μ satisfies the condition

$$u_\mu f^\mu = 0.$$

The components of the 4-vector f^μ can be expressed in terms of the ordinary vector of force $\vec{F} = \frac{d}{dt}(\vec{p})$.

We have

$$f^\mu = mc\,\frac{du^\mu}{ds} = \frac{\cancel{c}}{\cancel{c}\sqrt{1-v^2/c^2}}\cdot\frac{dp^\mu}{dt}$$

For the space components we obtain

$$f^i = \frac{1}{\sqrt{1-v^2/c^2}} \frac{dp^i}{dt} = \frac{F^i}{\sqrt{1-v^2/c^2}}$$

In order to find the expression for the time-component f^0 in terms of F let us calculate explicitely

$$\vec{F} = \frac{d}{dt} \frac{m\vec{v}}{\sqrt{1-v^2/c^2}} = m \left\{ \frac{\dot{\vec{v}}}{\sqrt{1-v^2/c^2}} + \frac{\vec{v}\,(\vec{v}\cdot\dot{\vec{v}})/c^2}{(1-v^2/c^2)^{3/2}} \right\}$$

$$= \frac{m}{\sqrt{1-v^2/c^2}} \left\{ \dot{\vec{v}} + \frac{\vec{v}\,(\vec{v}\cdot\dot{\vec{v}})/c^2}{1-v^2/c^2} \right\}$$

From this expression

$$(\vec{v}\cdot\vec{F}) = \frac{m\,(\vec{v}\,\dot{\vec{v}})}{\sqrt{1-v^2/c^2}} \left(1 + \frac{v^2/c^2}{1-v^2/c^2} \right) =$$

$$= \frac{m\,(\vec{v}\,\dot{\vec{v}})}{(1-v^2/c^2)^{3/2}}$$

Now, let us calculate

$$f^0 = \frac{1}{\sqrt{1-v^2/c^2}} \frac{dp^0}{dt} = \frac{mc}{\sqrt{1-v^2/c^2}} \frac{d}{dt} \left(\frac{1}{\sqrt{1-v^2/c^2}} \right)$$

$$= \frac{mc}{\sqrt{1 - v^2/c^2}} \cdot \frac{(\vec{v}\,\dot{\vec{v}})/c^2}{(1 - v^2/c^2)^{3/2}} = \frac{(\vec{F} \cdot \vec{v})}{c\sqrt{1 - v^2/c^2}}$$

Thus,

$$f^\mu = \left(\frac{(\vec{F} \cdot \vec{v})}{c\sqrt{1 - v^2/c^2}} , \frac{\vec{F}}{\sqrt{1 - v^2/c^2}} \right)$$

we see that the 0-th component of the 4-force is related to the work of the force.

<u>To summarize</u>: the relativistic generalization of the Newton's law, eq. (44*), is equivalent to the equation

$$\frac{d}{dt}(P^i) = F^i$$

where $P^i = \dfrac{mv^i}{\sqrt{1 - v^2/c^2}}$ is the relativistic momentum.

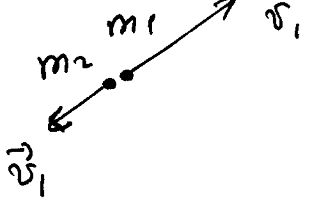
Application of energy-momentum conservation to particle decay and collisions

$$\sum_i P^\mu_{in} = \sum_f P^\mu_{fin}$$

1. Decay. Consider the decay of a massive particle into two particles. We can always choose the reference frame in such a way that the initial particle is at rest, $\vec{P}_{in} = 0$. The energy of the initial particle is then $P^0_{in} = Mc$, M being the mass of the initial particle.

$$\sum P^\mu_{in} = (Mc, 0).$$

Let final particles have velocities $\vec{v}_1$ and $\vec{v}_2$ and masses m_1 and m_2. The total momentum after the decay must be zero, and thus the directions of $\vec{v}_1$ and $\vec{v}_2$ must be opposite.



The conservation of the energy and momentum gives

$$Mc^2 = E_1 + E_2$$
$$0 = \underline{p}_1 - \underline{p}_2 \qquad (*)$$

There are also the following relations between energy and momentum of each particle:

$$E_1^2 - c^2 p_1^2 = c^4 m_1^2$$
$$E_2^2 - c^2 p_2^2 = c^4 m_2^2 . \qquad (**)$$

The second of eqs. (*) gives

$$p_1^2 = p_2^2$$

Subtracting eqs. (**) from one another gives:

$$E_1^2 - E_2^2 = c^4 (m_1^2 - m_2^2)$$

$$(E_1 - E_2)\underbrace{(E_1 + E_2)}_{c^2 M} = c^4 (m_1^2 - m_2^2)$$

$$\Rightarrow \quad \begin{cases} E_1 - E_2 = c^2 \dfrac{m_1^2 - m_2^2}{M} \\ E_1 + E_2 = Mc^2 \end{cases}$$

$$E_1 = \frac{1}{2} c^2 \left\{ M + \frac{m_1^2 - m_2^2}{M} \right\} =$$

$$\left\{ \begin{aligned} E_1 &= \frac{c^2}{2M} \left(M^2 + m_1^2 - m_2^2 \right) \\ E_2 &= \frac{c^2}{2M} \left(M^2 - m_1^2 + m_2^2 \right) \end{aligned} \right.$$

Corresponding momenta P_1 and P_2 can be obtained from (48 **).

Consistency check:

$$E_1 - c^2 m_1 = \frac{c^2}{2M} \left\{ M^2 + m_1^2 - m_2^2 - 2 M m_1 \right\} =$$

$$= \frac{c^2}{2M} \left\{ (M - m_1)^2 - m_2^2 \right\} =$$

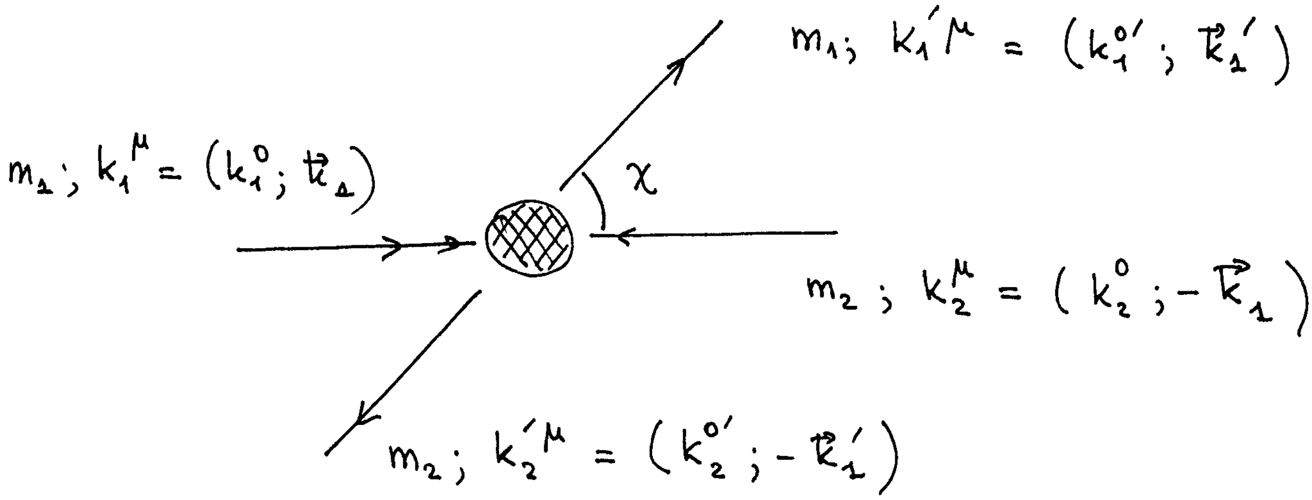
$$= \frac{c^2}{2M} (M - m_1 - m_2)(M - m_1 + m_2)$$

This expression is greater than zero as long as $M > m_1 + m_2$ $\Rightarrow$ Ok

Elastic collisions

The elastic collisions are those in which the nature of particles does not change, i.e. the initial and final states contain the same particles, but with different energies and momenta.

The elastic collision of two particles is easiest to describe in the center-of-mass frame, i.e. the frame where the total momentum is zero. In this frame the collision looks as follows:



→ index 1, 2 refers to particles
′ (prime) denotes quantities after collision.

The collision in CM frame looks simplest because the energies and absolute values of momenta of particles <u>do not change</u>. The only change is in the directions of momenta. Indeed, from energy and momentum conservation we have

$$\epsilon_1(k_1) + \epsilon_2(k_2) = \epsilon_1(k_1') + \epsilon_2(k_2')$$

$$k_1 = k_2$$

$$k_1' = k_2'$$

Thus, from the first equation

$$\epsilon_1(k_1) + \epsilon_2(k_1) = \epsilon_1(k_1') + \epsilon_2(k_1') \qquad (*)$$

Since the function $\epsilon_1(k) + \epsilon_2(k) \equiv \sqrt{k_1^2c^2 + m_1^2c^4} + \sqrt{k^2c^2 + m_2^2c^4}$ is monotonic, for a given k_1 eq. $(*)$ has only one solution

$$k_1' = k_1,$$

and therefore

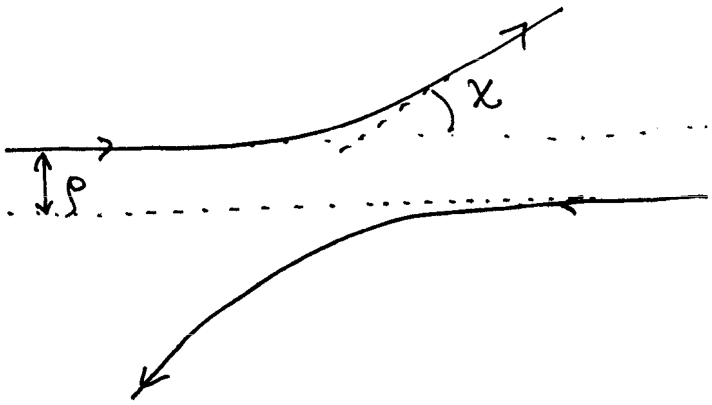
$$k_2' = k_2$$

$$\epsilon_1 = \epsilon_1'$$

$$\epsilon_2 = \epsilon_2'.$$

What determines the angle χ ? In real world particles interact by the fields they create (e.g., electric field). The interaction is expressed in terms of the <u>interaction potential</u> (e.g., electric potential).

The real collision looks as follows:

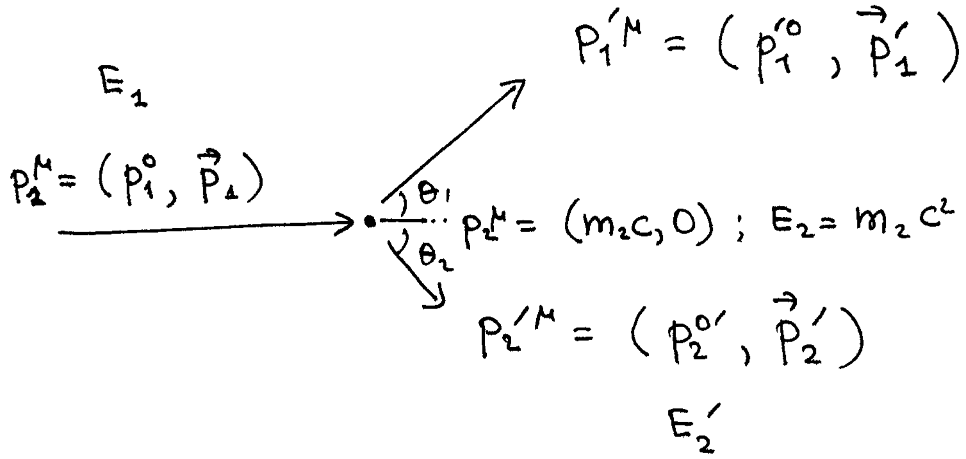


ρ = impact parameter.

The impact parameter determines χ. This relation involves the interaction potential $U(R)$, R being the distance between particles.

Knowing $\chi(\rho)$ one may determine $U(R)$.
This is the purpose of collision experiments.

The collision experiments are often made in such a way that one of the particles is at rest (fixed target). Let us express the energies of final particles (and angles θ_1, θ_2) in terms of the angle χ. In the lab. frame the collision looks as follows (assuming that 2 particle is at rest):



Conservation of 4-momentum gives:

$$p_1^\mu + p_2^\mu = p_1'^\mu + p_2'^\mu$$

This relation can be written as follows,

$$p_1^\mu + p_2^\mu - p_1'^\mu = p_2'^\mu$$

Square it (in the 4-d cense)

$$\overset{c^2 m_1^2}{(p_1 \cdot p_1)} + \overset{\cancel{c^2 m_2^2}}{(p_2 \cdot p_2)} + \overset{c^2 m_1^2}{(p_1' \cdot p_1')} + 2(p_1 p_2) -$$

$$- 2(p_2 p_1') - 2(p_1 p_1') = \underset{\cancel{c^2 m_2^2}}{(p_2' p_2')}$$

Thus we get

$$c^2 m_1^2 + (p_1 \cdot p_2) - (p_2 \cdot p_1') - (p_1 \cdot p_1') = 0 \qquad (*)$$

Similarly, we find

$$c^2 m_2^2 + (p_1 p_2) - (p_2 p_2') - (p_1 p_2') = 0 \qquad (**)$$

Making use of the fact that $p_2 = (m_2 c, 0)$ we find:

$$(p_1 p_2) = \frac{1}{c} \cdot E_1 \cdot m_2 c = m_2 E_1$$

$$(p_1' p_2) = \frac{1}{c} E_1' \, m_2 c = m_2 E_1'$$

$$(p_1 p_1') = \frac{1}{c^2} E_1 E_1' - p_1 p_1' \cos\theta_1$$

——— " ———

$$(p_2' p_2) = \frac{1}{c} E_2' \, m_2 c = m_2 E_2'$$

$$(p_1 p_2') = \frac{1}{c^2} E_1 E_2' - p_1 p_2' \cos\theta_2$$

Then from $(*)$ we find

$$-c^2 m_1^2 - m_2 E_1 + m_2 E_1' + \frac{1}{c^2} E_1 E_1' = p_1 p_1' \cos\theta_1$$

$$\Rightarrow \boxed{\cos\theta_1 = \frac{1}{p_1 p_1'} \left\{ \frac{1}{c^2} E_1 E_1' + m_2 E_1' - m_2 E_1 - c^2 m_1^2 \right\}}$$

Similarly, from (53 **) :

$$c^2 m_2^2 + m_2 E_1 - m_2 E_2' - \frac{1}{c^2} E_1 E_2' + p_1 p_2' \cos\theta_2 = 0$$

$$\cos\theta_2 = \frac{1}{p_1 p_2'} \left\{ \frac{1}{c^2} E_1 E_2' + m_2 E_2' - m_2 E_1' - m_2^2 c^2 \right\} =$$

$$\boxed{\cos\theta_2 = \frac{1}{p_1 p_2'} \left(\frac{1}{c} E_1 + c m_2 \right) \left(\frac{1}{c} E_2' - c m_2 \right)}$$

To express the energies E_1' and E_2' (and thus the angles θ_1 and θ_2) in terms of the angle χ consider again the relation (53*), but now note that the scalar products entering (53*) are Lorentz-invariant, so we can calculate $(p_1 \cdot p_1')$ in the CM frame. We have

$$(p_1 p_1') = (k_1 \cdot k_1') =$$

$$= \frac{1}{c^2} \epsilon_1 \epsilon_1' - k_1 \cdot k_1' \cdot \cos\chi =$$

$$= \frac{1}{c^2} \epsilon_1^2 - k_1^2 \cos\chi = \underbrace{m_1^2 c^2 + k_1^2}_{\frac{1}{c^2}\epsilon_1^2} - k_1^2 \cos\chi$$

$$= m_1^2 c^2 + k_1^2 (1 - \cos\chi)$$

Now use again the relation (53*):

$$\cancel{c^2 m_1^2} + m_2 E_1 - m_2 E_1' - \cancel{m_1^2 c^2} - k_1^2 (1-\cos\chi) = 0$$

$$\Rightarrow \quad E_1 - E_1' = \frac{k_1^2}{m_2}(1-\cos\chi)$$

Now we need to find k_1^2. To this end use the Lorentz-invariance of $(p_1 p_2) = (k_1 k_2)$:

$$m_2 E_1 = \frac{1}{c^2}\epsilon_1\epsilon_2 - k_1\cdot k_2 = \frac{1}{c^2}\epsilon_1\epsilon_2 + k_1^2$$

$$m_2 E_1 - k_1^2 = \sqrt{(c^2 m_1^2 + k_1^2)(c^2 m_2^2 + k_1^2)} \quad \left(\text{equation for } k_1^2\right)$$

$$m_2^2 E_1^2 + 2k_1^2 m_2 E_1 + \cancel{(k_1^2)^2} = c^4 m_1^2 m_2^2 + k_1^2 (m_1^2 + m_2^2) c^2 + \cancel{k_1^4}$$

$$k_1^2 = \frac{m_2^2 (E_1^2 - c^4 m_1^2)}{c^2 m_1^2 + c^2 m_2^2 + 2 m_2 E_1}$$

Therefore,

$$E_1' = E_1 - \frac{m_2 (E_1^2 - c^4 m_1^2)}{(m_1^2 + m_2^2)c^2 + 2m_2 E_1}(1-\cos\chi)$$

From energy conservation,

$$E_2' = c^2 m_2 + \frac{m_2 (E_1^2 - c^4 m_1^2)}{c^2 m_1^2 + c^2 m_2^2 + 2m_2 E_1}\cdot(1-\cos\chi)$$

These equations have the form

$$E_1' = E_1 - \Delta E$$

$$E_2' = \underset{\overset{\shortparallel}{E_2}}{m_2 c^2} + \Delta E$$

the energy transferred from particle 1 to particle 2

The maximum value of the transferred energy is

$$\Delta E_{max} = \frac{2 m_2 (E_1^2 - c^4 m_1^2)}{c^2 m_1^2 + c^2 m_2^2 + 2 m_2 E_1}$$

ELECTROSTATICS

* Coulomb's law: force acting on charge q_1

$$\vec{F} = k \, q_1 \cdot q_2 \cdot \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}$$

$$k = \frac{1}{4\pi\epsilon_0} \sim 8{,}988 \cdot 10^9 \cdot \frac{N \cdot m^2}{C^2}$$

Experimentally one measures the <u>force</u> acting on a small point charge q_1. This force is proportional to the charge q_1 when q_1 is small

$$\vec{F} = q_1 \vec{E}$$

where

$$\vec{E} = k \cdot q_2 \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}$$

Despite this way of introducing the electric field, we will see in what follows that the field E is a <u>material object</u> – something that propagates in space, carries energy & momentum, etc.

* Forces due to several charges add (linear superposition), and so do electric fields. Thus, the field created by several charges q_i equals

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} q_i \cdot \frac{\vec{x} - \vec{x}_i}{|\vec{x} - \vec{x}_i|^3} \qquad (*)$$

* Continuous distribution of charges is described by the charge density $\rho(x)$. By definition, the charge contained in a volume element $dV = dx^1 dx^2 dx^3 \equiv d^3x$ is

$$dq = \rho(x) \cdot dV = \rho(x) \cdot d^3x$$

In the case of continuous distribution of charge, eq. $(*)$ is written as

$$\vec{E}(x) = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \rho(\vec{x}') \cdot \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}$$

Math reminder

Let $f(\vec{x})$ be a smooth function of space and $\vec{g}(\vec{x}) = (g_1(\vec{x}), g_2(\vec{x}), g_3(\vec{x}))$ be a smooth vector field.

* $\vec{\nabla} f = (\partial_1 f, \partial_2 f, \partial_3 f) \equiv \left(\frac{\partial}{\partial x^1} f, \frac{\partial}{\partial x^2} f, \frac{\partial}{\partial x^3} f\right)$

$=$ gradient of f (vector!)

* $\vec{\nabla} \cdot \vec{g} = \partial_1 g_1 + \partial_2 g_2 + \partial_3 g_3 = \partial_i g_i$

$=$ divergence of $\vec{g}$ (scalar)

* $\vec{\nabla} \times \vec{g} = (\partial_2 g_3 - \partial_3 g_2, \partial_3 g_1 - \partial_1 g_3, \partial_1 g_2 - \partial_2 g_1)$

$=$ curl of $\vec{g}$ (vector!)

Equation
$$\vec{A} = \vec{\nabla} \times \vec{g}$$
can also be written in components as

$$A_i = \varepsilon_{ijk} \partial_j g_k$$
(recall: summation is implicit $= \sum_{jk} \varepsilon_{ijk} \partial_j g_k$)

where ε_{ijk} is totally antisymmetric tensor defined by

$$\varepsilon_{ijk} = -\varepsilon_{jik}$$

$$\varepsilon_{ijk} = -\varepsilon_{ikj}$$

$$\varepsilon_{123} = 1$$

* <u>Gauss' theorems</u>

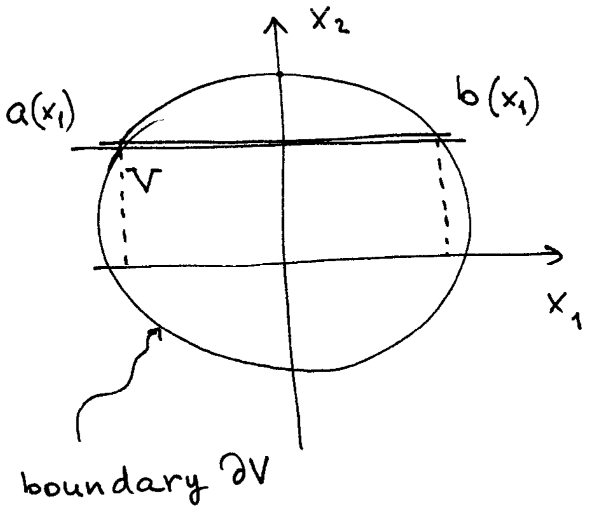
1d :

$$\int_a^b dx \cdot \partial_x f(x) = f(b) - f(a)$$

2d :

$$\int_V d^2x \cdot \partial_i f_i(x) \equiv \quad \text{(here } i = 1,2\text{)}$$

$$\equiv \int_V d^2x \ \vec{\nabla} \cdot \vec{f}(\vec{x}) = ?$$



Let us calculate:

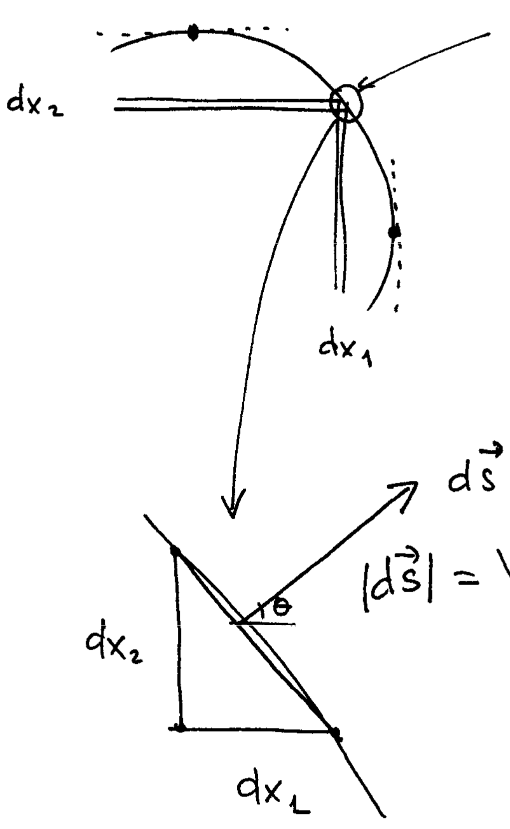
$$\int_V d^2x \left(\partial_1 f_1(x_1, x_2) + \partial_2 f_2(x_1, x_2) \right) =$$

$$= \int dx_2 \left[\int_{a(x_2)}^{b(x_2)} dx_1 \partial_1 f_1 \right] + \int dx_1 \left[\int_{\tilde{a}(x_1)}^{\tilde{b}(x_1)} dx_2 \partial_2 f_2 \right]$$

$$= \int dx_2 \left[f_1\left(b(x_2), x_2\right) - f_1\left(a(x_2), x_2\right) \right]$$

$$+ \int dx_1 \left[f_2\left(x_1, \tilde{b}(x_1)\right) - f_2\left(x_1, \tilde{a}(x_1)\right) \right] .$$

$\Rightarrow$ contributions come only from the boundary.



in this region only first terms contribute:

$$\int dx_2\, f_1 + \int dx_1\, f_2$$

$$|d\vec{s}| = \sqrt{dx_2^2 + dx_1^2}$$

Let $d\vec{s}$ be the vector orthogonal to the boundary with the length $ds = \sqrt{dx_1^2 + dx_2^2}$

$$d\vec{s} = ds \cdot (\cos\theta, \sin\theta)$$

$$\cos\theta = \frac{dx_2}{ds}$$

$$\sin\theta = \frac{dx_1}{ds}$$

$$\Rightarrow \boxed{d\vec{s} = (dx_2, dx_1)}$$

Our integral thus becomes

$$\int dx_2 \cdot f_1 + \int dx_1 \cdot f_2 = \int d\vec{s} \cdot \vec{f}$$

(over the boundary)

Other boundary parts give the same result.
Thus we obtain

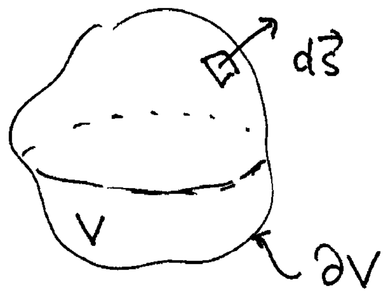
$$\int_V d^2x \; \vec{\nabla} \cdot \vec{f}(\vec{x}) \;=\; \int_{\partial V} d\vec{s} \cdot \vec{f}$$

3d: In 3 dimensions the story is the same (see proof in math books):

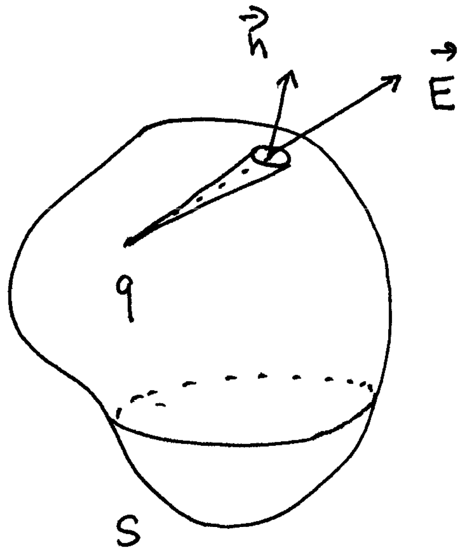
$$\int_V d^3x \; \vec{\nabla} \cdot \vec{f}(\vec{x}) \;=\; \int_{\partial V} \underbrace{d\vec{s}} \cdot \vec{f}$$

element of the surface ∂V:

- its direction is normal to the surface
- its value equals to the area element



* Gauss' law

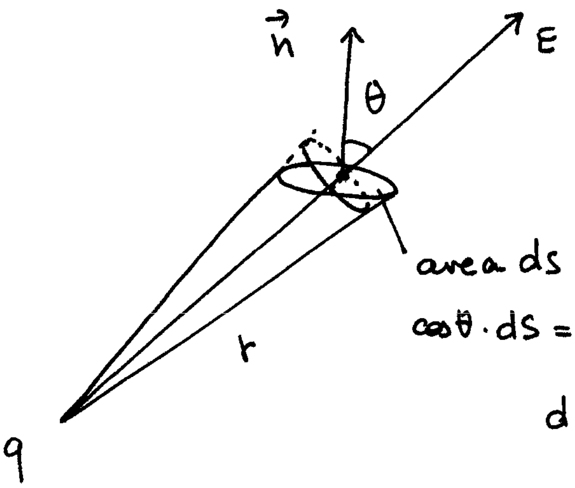


Consider closed surface S that includes a charge q

$\vec{E}$ - electric field

$\vec{n}$ - unit normal vector

$d\vec{s} = \underbrace{ds}_{\text{area element}} \cdot \vec{n}$ - surface element



$\cos\theta \cdot ds = r^2 d\Omega$

$d\Omega$ = solid angle of the cone

Now consider the integral

$$\int_S \vec{E} \cdot d\vec{s} = \int_S \underbrace{\frac{1}{4\pi\epsilon_0} \cdot q \cdot \frac{\cos\theta}{r^2}}_{\vec{E}\cdot\vec{n}} \cdot ds =$$

$$= \frac{q}{4\pi\epsilon_0} \int_S \frac{1}{\cancel{r^2}} \cdot \cancel{r^2} d\Omega = \frac{q}{\cancel{4\pi}\epsilon_0} \cdot \cancel{4\pi} = \frac{q}{\epsilon_0}$$

If there are many charges inside S, their contributions add. Thus

$$\int_S \vec{E}\cdot d\vec{s} = \sum_i \frac{q_i}{\epsilon_0}$$

sum over charges inside S

In the continuum limit:

$$\boxed{\int_S \vec{E}\cdot d\vec{s} = \frac{1}{\epsilon_0}\int_V \rho(x)\, d^3x}$$

use Gauss' theorem for this integral:

$$\int_S \vec{E}\cdot d\vec{s} = \int_V d^3x \cdot \underbrace{\vec{\nabla}\cdot\vec{E}}_{\text{divergence of } \vec{E}}$$

$$\Rightarrow \int_V d^3x\ \vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\int_V d^3x\ \rho(x)$$

Since the volume is arbitrary, we get

$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho}$$

(one of 4 Maxwell eqs!)

* Electrostatic potential

Go back to Columb law (58**) :

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \rho(\vec{x}') \cdot \underbrace{\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}}$$

this factor one can write as $-\vec{\nabla} \cdot \frac{1}{|\bar{x}-\bar{x}'|}$

Indeed, in components we have

$$- \partial_i \frac{1}{\sqrt{(x-x')_i (x-x')_i}} =$$

$$= - \left(-\frac{1}{2}\right) \frac{2(x-x')_i}{[(x-x')_i (x-x')_i]^{3/2}} = \frac{(x-x')_i}{|\vec{x}-\vec{x}'|^3} \quad \text{Ok}$$

Thus

$$\vec{E}(\vec{x}) = -\vec{\nabla} \left\{ \frac{1}{4\pi\epsilon_0} \int \rho(\vec{x}') \frac{d^3x'}{|\vec{x}-\vec{x}'|} \right\} = -\nabla \Phi \qquad (*)$$

where
$$\phi = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\bar{x}')}{|\bar{x} - \bar{x}'|} \, d^3x'$$

is the electrostatic potential, or <u>scalar potential</u>.

Taking curl of eq. (65*) we find

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times (\vec{\nabla}\phi) = 0$$

Therefore,
$$\vec{\nabla} \times \vec{E} = 0 .$$

This is (static part of) another Maxwell equation.

Finally, eqs
$$\vec{E} = -\vec{\nabla}\phi \quad , \quad \vec{\nabla}\vec{E} = \frac{\rho}{\epsilon_0}$$
imply
$$\vec{\nabla}\vec{\nabla}\phi \equiv \Delta\phi = -\frac{\rho}{\epsilon_0} \qquad \text{(Poisson eq.)}$$

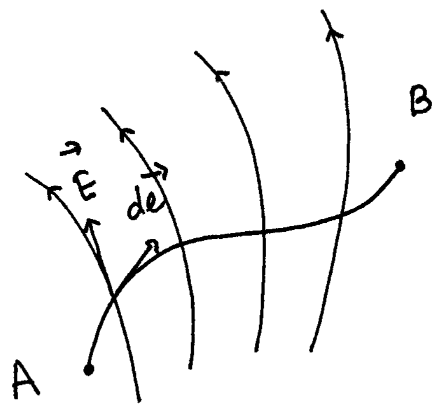
In the absence of charges this gives

$$\Delta\phi = 0 \qquad \text{(Laplace eq)}$$

↳ Laplacian $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$

* Interpretation of the potential Φ.

Consider slow displacement of charge q in the electric field from point A to point B.



Since force is

$$\vec{F} = q \cdot \vec{E}$$

the work of this force is

$$W = \int_A^B d\vec{l} \cdot \vec{F} = +q \int_A^B d\vec{l} \cdot \vec{E} =$$

$$= -q \int_A^B d\vec{l} \cdot \vec{\nabla}\phi = -q\left(\phi_B - \phi_A\right)$$

The change of particle energy is

$$-W = q\left(\phi_B - \phi_A\right)$$

$\Rightarrow$ Φ is a potential energy of charge in the electric field.

Usually one chooses ϕ in such a way that at infinity $\phi(\infty) = 0$.

* Potential energy. Capacitance

If a charge q_i is brought from infinity to a point x_i in a region of localized potential $\Phi(\bar{x})$, the work done on the charge is

$$W_i = q_i \Phi(\bar{x}_i)$$

If the potential is produced by $n-1$ charges,

$$\Phi(x_i) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{n-1} \frac{q_j}{|\bar{x}_i - \bar{x}_j|}$$

$$\Rightarrow \quad W_i = \frac{q_i}{4\pi\epsilon_0} \sum_{j=1}^{n-1} \frac{q_j}{|\bar{x}_i - \bar{x}_j|}$$

The total energy is thus:

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j<i} \frac{q_i q_j}{|x_i - x_j|} = \frac{1}{8\pi\epsilon_0} \sum_{i,j}{}' \frac{q_i q_j}{|x_i - x_j|}$$

In the case of a continuous distribution one has

$$W = \frac{1}{8\pi\epsilon_0} \int d\bar{x}\, d\bar{x}' \frac{\rho(x)\, \rho(x')}{|x - x'|} = \frac{1}{2} \int \rho(x)\, \Phi(x)\, dx \qquad (*)$$

One may interpret this energy in a different way, by making use of the Poisson equation

$$\rho(x) = -\epsilon_0 \, \Delta \Phi$$

so that

$$W = -\frac{\epsilon_0}{2} \int \phi(x) \, \Delta \, \phi(x) \, dx = \frac{\epsilon_0}{2} \int |\vec{\nabla} \phi|^2 \, dx =$$

$$= \frac{\epsilon_0}{2} \int |\vec{E}|^2 \, d^3x$$

$\Rightarrow$ The energy is expressed in terms of the electric field! The energy density in the region of the field $\vec{E}$ is thus

$$w = \frac{\epsilon_0}{2} |\vec{E}|^2$$

<u>Capacity</u>. For a system of n conductors, each having potential ϕ_i and charge Q_i, the electrostatic potential energy can be expressed in terms of potentials ϕ_i alone and certain geometrical coefficients called <u>capacities</u>. Since potentials are linear in charges, one may write

$$\phi_i = \sum_j p_{ij} \, Q_j$$

Inverting these relations one has

$$Q_i = \sum_j C_{ij} \Phi_j$$

The coefficients C_{ii} are called capacities.

C_{ij} at $i \neq j$ – coefficients of induction.

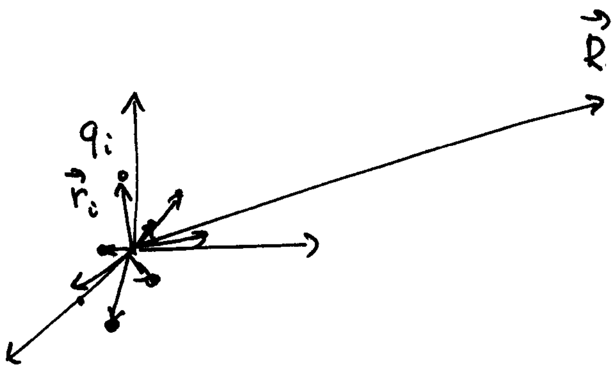
From (78*):

$$W = \frac{1}{2} \sum_i Q_i \Phi_i = \frac{1}{2} \sum_{i,j=1}^{n} C_{ij} \Phi_i \Phi_j$$

* Multipole expansion

In many situations the charges are distributed within some finite volume, and one needs to find the electrostatic potential (and, therefore, the electric field) at large distance from this region. In these cases a useful approximation can be obtained as follows.

Let charges q_i be placed in the points $\vec{r}_i$ which occupy some finite region of space (we choose the origin of the reference frame inside that region), and let $\vec{R}$ be the observation point such that $|\vec{R}| \gg |\vec{r}_i|$



The electrostatic potential of this charge distribution is

$$\Phi(\vec{R}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{|\vec{R} - \vec{r}_i|}$$

We need to calculate this expression at large $|\vec{R}|$.

Expansion of the denominator is done by means of the obvious relation

$$f(\vec{R}-\vec{r}) = f(\vec{R}) - r_i \cdot \frac{\partial f}{\partial R_i} + \frac{1}{2} r_i r_j \frac{\partial^2 f}{\partial R_i \partial R_j} + \ldots \qquad (*)$$

which is just the Taylor expansion. The first term is

$$f(\vec{R}-\vec{r}) = f(\vec{R}) - \vec{r} \cdot \vec{\nabla}_R f(\vec{R}) + \ldots$$

$$\frac{\partial}{\partial R_i} \frac{1}{|\vec{R}|} = - \frac{R_i}{|\vec{R}|^3} \quad \Rightarrow \quad \vec{\nabla} \frac{1}{R} = - \frac{\vec{R}}{R^3}$$

(here $R \equiv |\vec{R}|$)

Therefore, we have:

$$\Phi(\vec{R}) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{1}{|R|} \underbrace{\sum_i q_i}_{\substack{\text{total charge} \\ Q = \sum_i q_i}} - \underbrace{\sum_i q_i \vec{r}_i}_{\substack{\text{dipole moment} \\ \vec{d} = \sum q_i \vec{r}_i}} \cdot \frac{-\vec{R}}{R^3} + \ldots \right\}$$

Thus,

$$\Phi(\vec{R}) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{Q}{R} + \frac{\vec{d}\,\vec{R}}{R^3} + \ldots \right\}$$

It is important that if total charge of the system is zero, $Q = \sum_i q_i = 0$, then the dipole moment $\vec{d}$ is independent of the choice of the origin. Indeed, if we change the origin

$$\vec{r}_i \to \vec{r}_i + \vec{a},$$

then the dipole moment will change by

$$\vec{d} \to \sum_i q_i (\vec{r}_i + \vec{a}) = \vec{d} + \vec{a} \sum q_i = \vec{d},$$

i.e., $\vec{d}$ remains unchanged.

Consider the next order. We have from (84*):

$$\frac{\partial}{\partial R_i} \frac{1}{R} = - \frac{R_i}{R^3}$$

$$\frac{\partial^2}{\partial R_i \partial R_j} \frac{1}{R} = - \frac{\delta_{ij}}{R^3} + 3 \frac{R_i R_j}{R^5} = \frac{-R^2 \delta_{ij} + 3 R_i R_j}{R^5}$$

Thus

$$\frac{1}{|\bar{R} - \bar{r}|} = \frac{1}{|\bar{R}|} + \frac{\vec{r}\vec{R}}{R^3} + \frac{3(\bar{R}\bar{r})(\bar{R}\bar{r}) - \bar{R}^2 \bar{r}^2}{R^5}$$

Now we plug this expression into the scalar potential,

$$\Phi(\vec{R}) = \frac{1}{4\pi\epsilon_0} \cdot \left\{ \frac{Q}{R} + \frac{\vec{d}\,\vec{R}}{R^3} + \frac{1}{2} D_{ij} \frac{R_i R_j}{R^5} + \ldots \right\}$$

where we have introduced the <u>quadrupole moment</u>

$$D_{ij} \equiv \sum_k q^{(k)} \left(3 r_i^{(k)} r_j^{(k)} - \vec{r}^{(k)2} \cdot \delta_{ij} \right)$$

One may continue this expansion.

The expansion for the electrostatic potential Φ allows to build analogous expansion for the electric field $\vec{E}$: from $\vec{E} = -\vec{\nabla}\Phi$ we have

$$\vec{E}(\vec{R}) = \frac{1}{4\pi\epsilon_0} \left\{ + \frac{Q\vec{R}}{R^3} - \vec{\nabla} \frac{(\vec{d}\,\vec{R})}{R^3} + \ldots \right\}$$

$$= \frac{1}{4\pi\epsilon_0} \left\{ \frac{Q\vec{R}}{R^3} + d_i \frac{3R_iR_j - R^2\delta_{ij}}{R^5} + \ldots \right\}$$

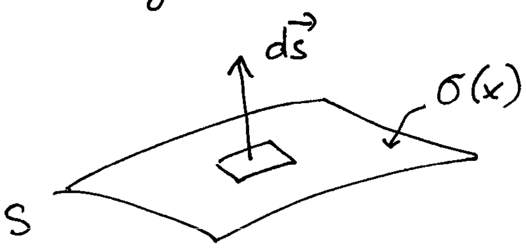
$$= \frac{1}{4\pi\epsilon_0} \cdot \left\{ \frac{Q\vec{R}}{R^3} + \frac{3\vec{R}\,(\vec{d}\cdot\vec{R}) - \vec{R}^2 \cdot \vec{d}}{R^5} + \ldots \right\}$$

this part is dominant at large R if $Q \neq 0$; it behaves like $\frac{1}{R^2}$

this part is the electric field of the dipole $\vec{d}$; it behaves like $\frac{1}{R^3}$

* <u>Surface distribution of charges. Continuity conditions for $\vec{E}$ and Φ across the surface.</u>

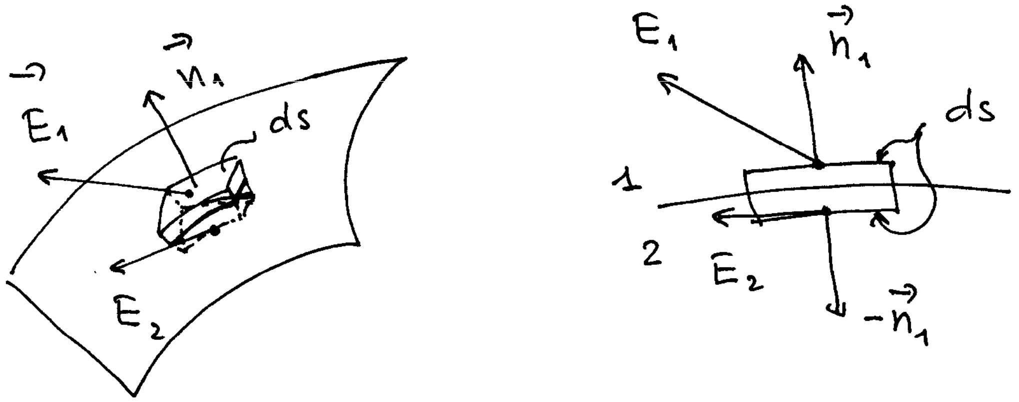
In practice there often arise charge distributions that are concentrated along certain <u>surface</u> (e.g., the boundary between two different materials). Such distributions are characterized by a <u>surface density</u> of charge $\sigma(x)$



$$dQ = \sigma(x) . ds$$

$\uparrow$ area element

Let us determine continuity conditions of $\vec{E}$ and Φ across the surface S carrying the surface charge density $\sigma(x)$.



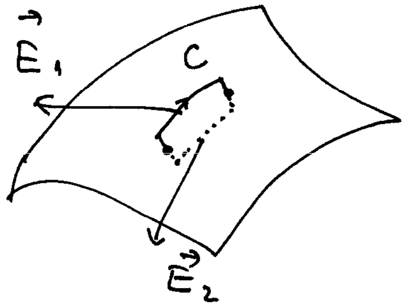
Consider a flat box of a very small thickness and area ds as shown on the figure. Gauss' law for this box reads (neglecting side contributions):

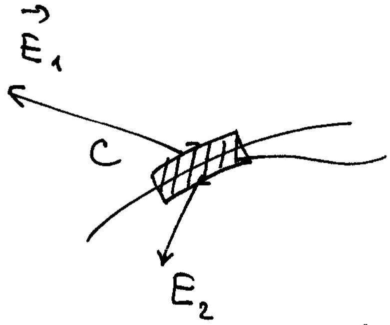
$$\vec{E}_1 \cdot \vec{n}_1 \cancel{ds} - \vec{E}_2 \cdot \vec{n}_1 \cancel{ds} = \frac{\sigma(x)\,\cancel{ds}}{\epsilon_0}$$

$$\boxed{(\vec{E}_1 - E_2) \cdot \vec{n}_1 = \frac{\sigma(x)}{\epsilon_0}} \qquad (*)$$

$\Rightarrow$ Normal component of $\vec{E}$ is discontinuous with the discontinuity equal by eq$(*)$.

Consider now the <u>contour</u> C





area S limited by the contour C

From the equation

$$\vec{\nabla} \times \vec{E} = 0$$

$$\Downarrow$$

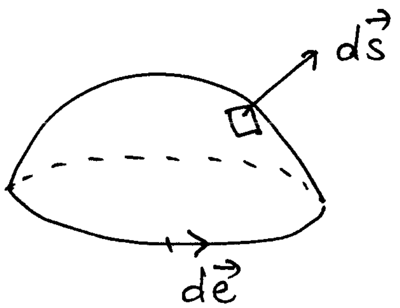
$$\int d\vec{s} \cdot (\vec{\nabla} \times \vec{E}) = 0$$

by Stokes theorem $\Rightarrow$

$$\int_C d\vec{l} \cdot \vec{E} = \boxed{\vec{E}_1 \cdot d\vec{l} - \vec{E}_2 d\vec{l} = 0}$$

$\Rightarrow$ tangent component of $\vec{E}$ is continuous.

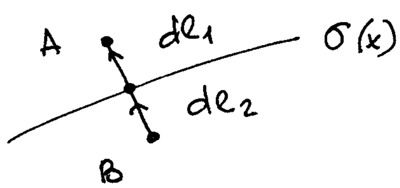
<u>Reminder</u>: Stokes theorem



$$\int d\vec{s} \cdot (\vec{\nabla} \times \vec{A}) = \int d\vec{l} \cdot \vec{A}$$

<u>Potential</u>

$$\phi = \int d\vec{l} \cdot \vec{E}$$

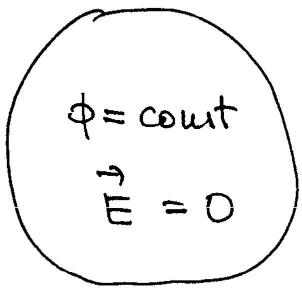


$$\phi_A - \phi_B = \vec{E}_1 \cdot d\vec{l}_1 + \vec{E}_2 \cdot d\vec{l}_2$$

$\Rightarrow$ $\phi_A - \phi_B \to 0$ when $A \to B$

$\Rightarrow$ $\boxed{\text{potential is \underline{continuous} across the surface}}$

<u>Example</u> : conductor. By definition, charges can move freely inside perfect conductors. Therefore, there cannot be a potential difference between two points of a conductor.



Outside of a conductor near its surface:

$E_{\parallel} = 0$ by continuity of tangential component

$\Rightarrow \vec{E} \parallel \vec{n}$, i.e. electric field is normal to the surface.

Note: charges may accumulate at the surface of a conductor. By the discontinuity equation we have

$$|\vec{E}| = \frac{\sigma(x)}{\epsilon_0}$$

where $\vec{E}$ is electric field just outside of the conductor.

* Poisson and Laplace equations

Electric potential satisfies a closed second-order equation

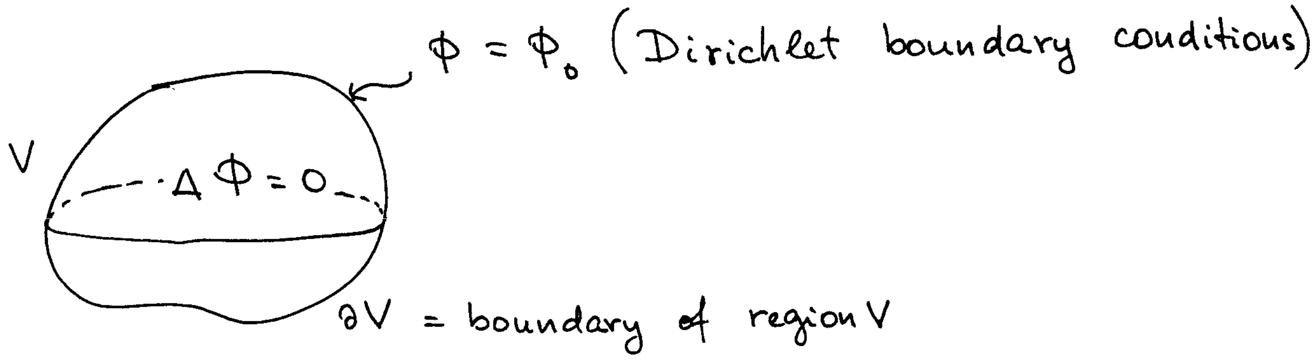
$$\Delta \phi = - \frac{\rho(x)}{\epsilon_0} \qquad \text{(Poisson equation)}$$

wher $\Delta \equiv \vec{\nabla} \cdot \vec{\nabla} = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator. At $\rho(x) = 0$ (no charges) we get

$$\Delta \phi = 0 \qquad \text{(Laplace equation)}$$

These equations are very common in physics, so we will descuss methods of their solution in general terms.

Uniqueness of the solution with Dirichlet boundary conditions



$$\begin{cases} \Delta \phi = - \dfrac{\rho(x)}{\epsilon_0} \\ \phi|_{\partial V} = \phi_0 \end{cases}$$

we want to show that the solution to this problem is unique

First, let us prove two useful identities known as Green's identities. Let $\phi(x)$ and $\psi(x)$ be two scalar fields. One has

$$\vec{\nabla} \cdot (\phi \vec{\nabla} \psi) = \vec{\nabla}\phi \cdot \vec{\nabla}\psi + \phi \Delta \psi$$

Now integrate this equation over the volume V,

$$\int_V d^3x \left\{ \vec{\nabla}\phi \cdot \vec{\nabla}\psi + \phi \Delta \psi \right\} = \int_V d^3x \, \vec{\nabla} \cdot (\phi \vec{\nabla} \psi) =$$

$$= \int_{\partial V} \phi \vec{\nabla} \psi \cdot d\vec{s}$$

Here $d\vec{s} = \vec{n} . ds$

$\vec{n} . \vec{\nabla} \psi \equiv \dfrac{\partial \psi}{\partial n}$ = normal derivative of ψ at the boundary $S = \partial V$

Thus,

$$\boxed{\int_V d^3x \left\{ \vec{\nabla}\phi \cdot \vec{\nabla}\psi + \phi \Delta \psi \right\} = \int_S ds \, \phi \cdot \frac{\partial \psi}{\partial n}} \qquad (*)$$

(First Green's identity)

Obviously, in the same way we have

$$\int_V d^3x \left\{ \vec{\nabla}\psi \cdot \vec{\nabla}\phi + \psi \Delta \phi \right\} = \int_S ds \, \psi \frac{\partial \varphi}{\partial n}$$

The second identity is obtained by taking the first one and subtracting the same with ϕ and ψ interchanged. One gets:

$$\int_V \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right) d^3x = \int_{\partial V} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) ds$$

Uniqueness of the solution with Dirichlet boundary conditions

We want to show that the solution of the Poisson equation $\vec{\nabla}^2 \Phi = \Delta \Phi = -\frac{\rho}{\epsilon_0}$ is unique for a given boundary value of potential specified at the boundary surface. ∂V of a given volume V (Dirichlet boundary condition).

Suppose that there exist two different solutions, Φ_1 and Φ_2. Denote

$$U = \Phi_2 - \Phi_1$$

Take the first Green's identity with $\phi = \psi = U$:

$$\int_V d^3x \left(U \Delta U + \vec{\nabla} U \cdot \vec{\nabla} U \right) = \int_{\partial V} U \cdot \frac{\partial U}{\partial n} \, ds$$

Since $\Delta U = 0$ and $U\big|_{\partial V} = 0$, this reduces to the equation

$$\int_V d^3x \, |\vec{\nabla} U|^2 = 0$$

$$\Rightarrow \quad \vec{\nabla} U = 0 \quad \Rightarrow \quad U \text{ is a constant.}$$

Since $U = 0$ at the boundary, $U = 0$ everywhere.

Note: the same argument applies to the case when we specify $\frac{\partial \phi}{\partial n}$ at the boundary. This type is called Neuman boundary conditions.

* Solution of Poisson and Laplace equations

In fact, we already have a solution to the Poisson equation,

$$\phi(x) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(x')}{|x - x'|} \, d^3x'$$

It is instructive to check explicitely that this is a solution. For this we need to introduce the Dirac δ-function.

* Dirac δ-function is defined by the relations

$$\delta(x-x_0) = 0 \quad \text{for } x \neq x_0$$

$$\int dx \; \delta(x-x_0)\cdot f(x) = f(x_0)$$

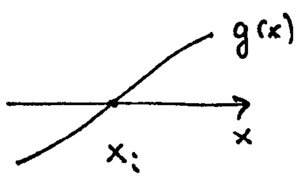
where $f(x)$ is a smooth function.

Derive a useful relation:

$$\delta\big(g(x)\big) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|} \qquad (*)$$

where x_i are roots of the equation $g(x)=0$.

$$\int dx\, f(x)\, \delta\big(g(x)\big) = \sum_i \int dx\, f(x_i)\cdot \delta\big(g'(x_i)(x-x_i)+\ldots\big)$$



(change of variables $y = g'(x_i)(x-x_i)$)

$$= \sum_i f(x_i) \int dy \frac{\delta(y)}{|g'(x_i)|} = \sum_i \frac{f(x_i)}{|g'(x_i)|}$$

$\Rightarrow$ we have shown (*).

* In 3 dimensions, one generalizes $\delta(x) \to \delta^3(x) = \delta(x_1)\,\delta(x_2)\,\delta(x_3)$, so that

$$\int d^3x\, f(\vec{x})\; \delta^3(\vec{x}-\vec{x}_0) = f(\vec{x}_0)$$

Now let us derive a very useful relation

$$\Delta \frac{1}{r} = -4\pi\, \delta^3(\vec{x}) \qquad \left(\text{here } r = |x| = \sqrt{x_i x_i}\right)$$

Let us calculate the l.h.s.:

$$\Delta \frac{1}{r} = \partial_i \partial_i \frac{1}{r} = \partial_i \left(- \frac{\partial_i r}{r^2} \right) \overset{*}{=}$$

$$\partial_i r = \partial_i \sqrt{x_j x_j} = \cancel{\tfrac{1}{2}} \frac{\cancel{2} x_i}{\sqrt{x_j x_j}} = \frac{x_i}{r}$$

$$\overset{*}{=} -\partial_i \left(\frac{x_i}{r^3} \right) =$$

$$= - \frac{\delta_{ii}}{r^3} + 3 \frac{x_i x_i}{r^5} = -\frac{3}{r^3} + \frac{3}{r^3} = 0 \quad !$$

$\Rightarrow$ $\Delta \frac{1}{r} = 0$ everywhere except the origin, where it is singular.

Now let us see that $\Delta \frac{1}{r}$ is proportional to a δ-function $\delta^3(\vec{x})$. To this end let us calculate the integral of $\Delta \frac{1}{r}$ with any smooth function $f(\vec{x})$.

$$\int_V d^3x \; f(\vec{x}) \cdot \Delta \frac{1}{r}$$

over the volume V which involves the origin $\vec{x} = 0$.

Since $\Delta \frac{1}{r}$ is zero everywhere except the origin, we may limit the integration region by the sphere of the small radius ε. Then the function $f(\vec{x})$ can be replaced by the constant $f(0)$.

$$\int_V d^3x \; f(\vec{x}) \cdot \Delta \frac{1}{r} = f(0) \cdot \int_\varepsilon d^3x \; \Delta \frac{1}{r} =$$

$$= f(0) \cdot \int_{\partial\varepsilon} \underbrace{\vec{\nabla} \frac{1}{r}}_{= -\frac{1}{r^2} \cdot \frac{\vec{x}}{r} = -\frac{\vec{n}}{r^2}} \cdot d\vec{s} = f(0) \cdot (-1) \int_{\partial\varepsilon} \frac{\vec{n}}{r^2} \, d\vec{s} =$$

$$= f(0) \cdot (-1) \frac{1}{\cancel{\varepsilon^2}} \cdot 4\pi \cancel{\varepsilon^2} = -4\pi f(0).$$

Therefore, we get

$$\boxed{\Delta \frac{1}{r} = -4\pi \, \delta^3(\vec{x})}$$

It follows then that

$$\Delta \frac{1}{|x-x'|} = -4\pi \, \delta^3(\vec{x}-\vec{x}').$$

With this relation we can show that (70 *) is a solution to the Poisson equation (70 v).

$$\Delta \Phi(x) = \frac{1}{4\pi\epsilon_0} \int_V \rho(x') \, \Delta \frac{1}{|x-x'|} \, d^3x' =$$

$$= -\frac{1}{\epsilon_0} \int \rho(x') \, \delta^3(\bar{x}-\bar{x}') \, d^3x' = -\frac{\rho(x)}{\epsilon_0} \quad \text{Ok}$$

Method of Green's function

The way we have found the solution (70 *) to the Poisson equation is known as the method of Green's function. Consider it in a more general framework. The equation that we need to solve is

$$\Delta \Phi = -\frac{\rho(x)}{\epsilon_0} \qquad (*)$$

We are looking for solutions in a given volume V with certain boundary conditions, We will consider as such boundary condition the value of Φ at ∂V.

Many equations in physics have similar form.

Suppose that we have found a solution to the equation

$$\Delta_x \, G(\bar{x}, \bar{x}') = \delta(\bar{x} - x') \qquad (*)$$

In fact, we know such a solution:

$$G(\bar{x}, \bar{x}') = -\frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{x}'|}$$

Then we can write a solution to the Poisson equation (75) :

$$\Phi(x) = \frac{-1}{\epsilon_0} \int_V dx' \cdot G(\bar{x}, \bar{x}') \cdot \rho(x')$$

Indeed, acting on $\Phi(x)$ by the operator Δ_x we find:

$$\Delta_x \Phi(x) = -\frac{1}{\epsilon_0} \int_V dx' \; \Delta_x G(\bar{x}, \bar{x}') \, \rho(x') =$$

$$= -\frac{1}{\epsilon_0} \int_V dx' \; \delta(\bar{x} - x') \, \rho(x')$$

$$= -\frac{\rho(x)}{\epsilon_0} \qquad \text{Ok}.$$

The problem remains to satisfy boundary conditions.

First, we note that eq. (76x) does not specify the function $G(\bar{x},\bar{x}')$ uniquely. To any solution we can add a function $F(\bar{x},\bar{x}')$ such that

$$\Delta_x F(x,x') = 0$$

[In math terms, general solution of the non-homogeneous equation is a sum of a particular solution of the inhomogeneous equation and general solution of the homogeneous equation]. One has

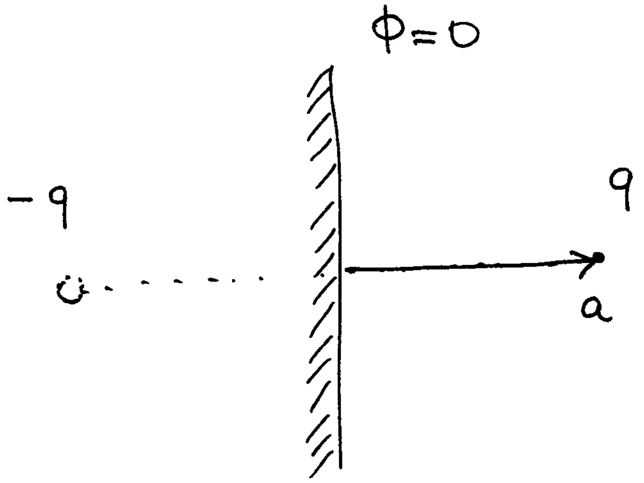
$$\Delta_x \Big(G(x,x') + F(x,x') \Big) = \delta(x-x')$$

The functions $\tilde{G}(\bar{x},\bar{x}') = G(\bar{x},\bar{x}') + F(\bar{x},\bar{x}')$ and $G(\bar{x},\bar{x}')$ differ by the boundary conditions. Thus, we can use this freedom to satisfy the desired boundary conditions.

The method of images

The method of images is suitable for problems of finding the electric field and potential of one or more charges in the presence of boundary surfaces, for instance conductors either grounded or held at fixed potentials. It consists in placing "image charges" outside of the region of interest in such a way as to satisfy the boundary conditions at the surfaces. In terms of Poisson equation, these image charges provide a solution to the homogeneous equation, while a particular solution to the inhomogeneous equation is given by the potential created by the real charges.

* The simplest example is provided by a grounded conducting plane (grounded = having zero potential) and a point charge



Conducting plane sets the boundary condition $\phi = 0$ at $x = 0$. Since the potential depends on the distance only and is negative for negative charges, one may satisfy the condition $\phi = 0$ by putting the charge $-q$ symmetrically with respect to the plane. Then whether the plane is present or not changes nothing.

The potential equals

at $x > 0$:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\vec{r}-\vec{a}|} - \frac{q}{|\vec{r}+\vec{a}|}\right)$$

clearly $\phi(0, y, z) = 0$ for all y, z

Of course, the "image charge" does not exist; its role is played by the surface charge density on the conducting plane. To find this charge density let us first calculate the electric field near the surface. Since we know the potential, we just make use of the relation

$$\vec{E} = -\vec{\nabla}\phi.$$

$$\vec{\nabla}\frac{1}{|\vec{r}-\vec{a}|} = -\frac{\vec{r}-\vec{a}}{|\vec{r}-\vec{a}|^3}$$

Thus,
$$E = -\vec{\nabla}\phi = \frac{q}{4\pi\epsilon_0}\left(\frac{\bar{r}-\bar{a}}{|\bar{r}-\bar{a}|^3} - \frac{\vec{r}+\vec{a}}{|\vec{r}+\vec{a}|^2}\right)$$

In the points of the plane we have

$$|\vec{r}-\vec{a}| = |\vec{r}+\vec{a}| = \sqrt{r^2+a^2}$$

Thus,
$$\vec{E} = -\frac{2q\vec{a}}{4\pi\epsilon_0 (r^2+a^2)^{3/2}}$$

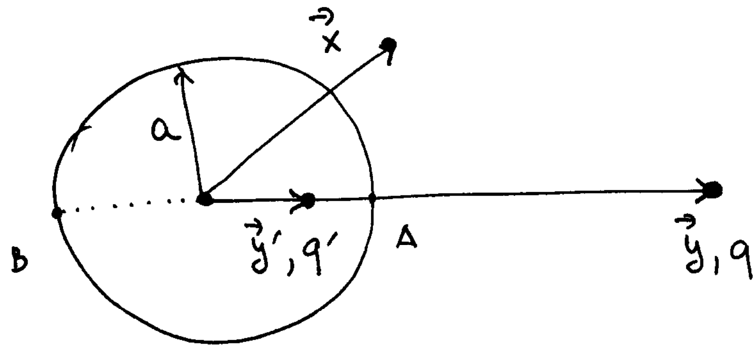
By the Gauss' theorem $\quad \bar{E}\, d\vec{s} = \dfrac{\sigma\, ds}{\epsilon_0}$

$$-\frac{2qa}{4\pi\epsilon_0 (r^2+a^2)^{3/2}} = \frac{\sigma}{\epsilon_0}$$

$$\Rightarrow \quad \sigma(r) = -\frac{qa}{2\pi (r^2+a^2)^{3/2}}$$

Note: dimension is OK

* <u>Another example</u>: point charge in the presence of a grounded conducting sphere



we have for the potential at $\vec{x}$:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \cdot \left\{ \frac{q}{|\vec{x} - \vec{y}|} + \frac{q'}{|\vec{x} - \vec{y}'|} \right\}$$

We must now choose q' and y' in such a way that the potential vanishes at $|\vec{x}| = a$.

Since this has to be correct everywhere on the sphere, it has to be correct at the points A and B as well. Thus, we have:

$$A: \quad \frac{q'}{a - y'} + \frac{q}{y - a} = 0$$

$$B: \quad \frac{q'}{y' + a} + \frac{q}{y + a} = 0$$

$$q'(y-a) + q(a-y') = 0$$

$$q'(y+a) + q(y'+a) = 0$$

adding and subtracting and dividing by 2

$$q'y + qa = 0$$

$$q'a + qy = 0 .$$

$$\boxed{\begin{aligned} q' &= -q\frac{a}{y} \\ y' &= -\frac{q'}{q}a = \frac{a^2}{y} \end{aligned}}$$

Now we have to check that $\phi = 0$ is satisfied in other points of the sphere. Introducing the notations

$$\vec{x} = \vec{n}\cdot|\vec{x}| \equiv x\cdot\vec{n}$$

$$\vec{y} = \vec{m}\cdot|\vec{y}| = y\,\vec{m}$$

we can write the potential as

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{|x\cdot\vec{n} - y\cdot\vec{m}|} + \frac{q'}{|x\cdot\vec{n} - y'\cdot\vec{m}|}\right\};$$

$$\Phi(a\cdot\vec{n}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{|a\vec{n} - y\cdot\vec{m}|} + \frac{q'}{|a\vec{n} - y'\vec{m}|}\right\}$$

$$= \frac{q}{4\pi\epsilon_0} \left\{ \frac{1}{|a\vec{n} - y\vec{m}|} - \frac{1}{y/a \cdot |a\vec{n} - \frac{a^2}{y}\vec{m}|} \right\} =$$

$$= \frac{q}{4\pi\epsilon_0} \left\{ \frac{1}{|a\vec{n} - y\vec{m}|} - \frac{1}{|y\vec{n} - a\vec{m}|} \right\}$$

The difference in the curled brackets vanishes:

$$|a\vec{n} - y\vec{m}| = \sqrt{a^2 + y^2 - 2ay(\bar{n}\bar{m})}$$

$$|y\vec{n} - a\vec{m}| = \sqrt{y^2 + a^2 - 2ay(\bar{n}\bar{m})}$$

Therefore, $\Phi(x=a) = 0$.

There are other applications of the method of images. However, in general it is limited to simple geometries.

Separation of variables

A powerful technique to solve partial differential equations (e.g. Laplace equation) is based on separation of variables.

Math reminder: orthogonal systems of functions.

Consider an infinite set of functions $U_n(x)$, $n = 1, 2, \ldots$ which are square integrable on the interval $x \in [a, b]$ and orthonormal:

$$\int_a^b dx \; U_n^*(x)\, U_m(x) = \delta_{mn}$$

This set of functions is said to be complete if

$$\sum_{n=1}^{\infty} U_n^*(x')\, U_n(x) = \delta(x - x')$$

[To prove completeness of a given set is usually a difficult math problem].

Any square-integrable function $f(x)$ can be represented as a series in functions $U_n(x)$:

$$f(x) = \sum_{n=1}^{\infty} a_n U_n(x)$$

where coefficients a_n can be found as follows:

$$\int_a^b U_m^*(x)\, f(x)\, dx = \int_a^b U_m^*(x) \left\{ \sum_{n=1}^{\infty} a_n U_n(x) \right\} dx =$$

$$= \sum_{n=1}^{\infty} a_n \int_a^b U_m^*(x) U_n(x)\, dx =$$

$$= \sum_{n=1}^{\infty} a_n \delta_{nm} = a_m$$

The necessity of the completeness condition follows from the following calulation:

$$f(x) = \sum_{n=1}^{\infty} a_n U_n(x) = \sum_{n=1}^{\infty} \left\{ \int_a^b dy\; U_n^*(y) f(y) \right\} U_n(x)$$

$$= \int dy \left\{ \sum_{n=1}^{\infty} U_n^*(y) U_n(x) \right\} f(y) . \qquad \stackrel{*}{=}$$

$\Rightarrow$ the quantity in the curl brackets must be a δ-function; then

$$\stackrel{*}{=} \int dy\; \delta(y-x) f(y) = f(x) \qquad \text{OK}$$

* Example of expansions in orthogonal set of functions - Fourier series.

Consider now the separation of variables in the Laplace equation, first in the simplest case of Cartesian coordinates. In these coordinates the Laplace equation reads

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad (*)$$

The idea of the method is to look for the solution in the form

$$\phi(x,y,z) = U(x)\cdot V(y)\cdot W(z)$$

Substituting into eq. $(*)$ and dividing by ϕ we get

$$\frac{1}{U(x)}\frac{d^2U}{dx^2} + \frac{1}{V(y)}\frac{d^2V}{dy^2} + \frac{1}{W(z)}\frac{d^2W}{dz^2} = 0$$

This equation can be satisfied <u>only</u> if each of the terms is separately a constant (one positive, two negative). Thus, we have traded a partial differential equation for 3 ordinary differential equations

$$\frac{1}{U(x)} \cdot \frac{d^2 U}{dx^2} = -\alpha^2$$

$$\frac{1}{V(y)} \cdot \frac{d^2 V}{dy^2} = -\beta^2$$

$$\frac{1}{W(z)} \frac{d^2 W}{dz^2} = \gamma^2 = \alpha^2 + \beta^2$$

The solutions to these equations are evident:

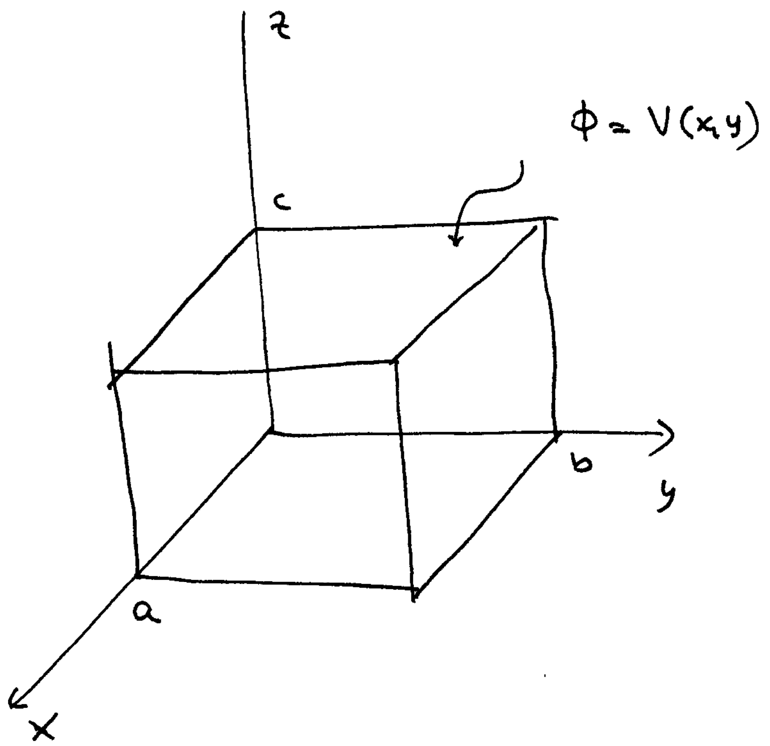
$$U(x) = e^{\pm i\alpha x}$$

$$V(y) = e^{\pm i\beta y}$$

$$W(z) = e^{\pm \gamma z} \qquad \text{where } \gamma = \sqrt{\alpha^2+\beta^2}$$

The coefficients α, β (and thus γ) are determined by the boundary conditions.

As an example, consider the problem of finding the electrostatic potential inside a rectangular box of dimensions a, b, c with zero boundary conditions everywhere except the surface $z = c$ where the potential is $V(x, y)$.



In order to have $\Phi = 0$ at $x=0$, $y=0$ and $z=0$ we have to choose

$$U(x) = \sin(\alpha x)$$
$$V(y) = \sin(\beta y)$$
$$W(z) = sh(\gamma z) \; ; \qquad \gamma = \sqrt{\alpha^2+\beta^2}$$

To have $\Phi = 0$ at $x = a$ and $y = b$ we must choose

$$\alpha_n = \frac{\pi n}{a} \qquad m, n = 1, 2, \dots$$
$$\beta_m = \frac{\pi m}{b}$$

and, therefore,

$$\gamma_{mn} = \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

The solution, satisfying all boundary conditions except one is, therefore

$$\Phi(x,y,z) = \sum_{m,n=1}^{\infty} A_{mn} \cdot \sin(\alpha_n x) \cdot \sin(\beta_m y) \cdot \operatorname{sh}(\gamma_{mn} z)$$

Now we have to choose A_{mn} to satisfy the remaining condotion at $z = c$:

$$V(x,y) = \Phi(x,y,c) =$$

$$= \sum_{m,n=1}^{\infty} A_{mn} \sin(\alpha_n x) \sin(\beta_m y) \cdot \operatorname{sh}(\gamma_{mn} \cdot c)$$

$$\Rightarrow \quad A_{mn} = \frac{4}{a\, b\, \operatorname{sh}(\gamma_{mn} c)} \times$$

$$\times \int_0^a dx \int_0^b dy\, V(x,y) \sin(\alpha_n x) \sin(\beta_m y)$$

$\Rightarrow$ the coefficients A_{mn} are determined by the coefficients of the double Fourier series for $V(x,y)$.

The solution with arbitrary potentials at all boundaries can be obtained as a linear superposition of such solutions

Separation of variables. Cylindrical coordinates

Cylindrical coordinates are defined by the relations

$$x = \rho \cos\varphi$$
$$y = \rho \sin\varphi$$

From these relations we find

$$dx = d\rho \cdot \cos\varphi - \rho \sin\varphi \, d\varphi$$
$$dy = d\rho \cdot \sin\varphi + \rho \cos\varphi \, d\varphi$$

and inverse relations

$$d\rho = \cos\varphi \cdot dx + \sin\varphi \cdot dy \equiv \frac{\partial\rho}{\partial x} dx + \frac{\partial \rho}{\partial y} dy$$
$$d\varphi = -\frac{1}{\rho}\sin\varphi \, dx + \frac{1}{\rho}\cos\varphi \, dy \equiv \frac{\partial\varphi}{\partial x} dx + \frac{\partial\varphi}{\partial y} dy$$

Therefore,

$$\partial_x = \frac{\partial\rho}{\partial x}\partial_\rho + \frac{\partial\varphi}{\partial x}\partial_\varphi = \cos\varphi \cdot \partial_\rho - \frac{1}{\rho}\sin\varphi \, \partial_\varphi$$
$$\partial_y = \frac{\partial\rho}{\partial y}\partial_\rho + \frac{\partial\varphi}{\partial y}\partial_\varphi = \sin\varphi \, \partial_\rho + \frac{1}{\rho}\cos\varphi \, \partial_\varphi$$

$$\partial_x^2 = \left(\cos\varphi\, \partial_\rho - \frac{1}{\rho}\sin\varphi\, \partial_\varphi\right)\left(\cos\varphi\, \partial_\rho - \frac{1}{\rho}\sin\varphi\, \partial_\varphi\right) =$$
$$= \cos^2\varphi\, \partial_\rho^2 + \frac{1}{\rho^2}\cos\varphi \sin\varphi\, \partial_\varphi - \frac{1}{\rho}\cos\varphi\sin\varphi\, \partial_\rho\partial_\varphi$$
$$+ \frac{1}{\rho^2}\sin\varphi\cos\varphi\, \partial_\varphi + \frac{1}{\rho}\sin^2\varphi\, \partial_\rho - \frac{1}{\rho}\sin\varphi\cos\varphi\, \partial_\rho\partial_\varphi + \frac{1}{\rho^2}\sin^2\varphi\, \partial_\varphi^2$$

$$\partial_y^2 = \left(\sin\varphi \cdot \partial_\rho + \frac{1}{\rho}\cos\varphi\, \partial_\varphi\right)\left(\sin\varphi\, \partial_\rho + \frac{1}{\rho}\cos\varphi\, \partial_\varphi\right) =$$

$$= \sin^2\varphi \cdot \partial_\rho^2 - \frac{1}{\rho^2}\sin\varphi\cos\varphi\, \partial_\varphi + \frac{1}{\rho}\sin\varphi\cos\varphi\, \partial_\rho \partial_\varphi$$

$$+ \frac{1}{\rho}\cos^2\varphi\, \partial_\rho + \frac{1}{\rho}\sin\varphi\cos\varphi\, \partial_\rho\partial_\varphi - \frac{1}{\rho^2}\sin\varphi\cos\varphi\, \partial_\varphi$$

$$+ \frac{1}{\rho^2}\cos^2\varphi\, \partial_\varphi^2$$

$$\partial_x^2 + \partial_y^2 = \left(\sin^2\varphi + \cos^2\varphi\right)\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\varphi^2$$

$$= \partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\varphi^2$$

$$= \frac{1}{\rho}\partial_\rho\left(\rho\,\partial_\rho\right) + \frac{1}{\rho^2}\partial_\varphi^2$$

Thus, in cylindrical coordinates the laplacian of Φ reads

$$\boxed{\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right)\Phi = \frac{1}{\rho}\partial_\rho\left(\rho\,\partial_\rho\Phi\right) + \frac{1}{\rho^2}\partial_\varphi^2\Phi + \partial_z^2\Phi}$$

Now we can discuss separation of variables in these coordinates. Let us look for the solution in the form

$$\Phi(\rho, \varphi, z) = R(\rho) \cdot \Psi(\varphi) \cdot W(z)$$

Substituting this into equation (101 *) we get the Laplace equation

$$\frac{1}{R} \cdot \frac{1}{\rho} \partial_\rho (\rho \partial_\rho R) + \frac{1}{\rho^2} \underbrace{\frac{1}{\Psi} \partial_\phi^2 \Psi}_{-n^2} + \underbrace{\frac{1}{W} \partial_z^2 W}_{k^2} = 0$$

this must be periodic in φ, so one must have

$$\Psi = e^{\pm i n \varphi}$$

(assuming the solution covers the whole space around $r = 0$, i.e, there is no sector cut away).

$$W = e^{\pm k z}$$

Thus, for $R(\rho)$ we get the following equation:

$$\partial_\rho^2 R + \frac{1}{\rho}\partial_\rho R + \left(k^2 - \frac{n^2}{\rho^2}\right) R = 0$$

This equation is called Bessel equation. As any second order equation, it has two solutions. This equation is brought to the standard form by defining a new variable $x = k\rho$:

$$\partial_x^2 R + \frac{1}{x}\partial_x R + \left(1 - \frac{\nu^2}{x^2}\right) R = 0 \qquad (*)$$

[We have assumed a general case $\nu \neq$ integer]

At $x \to 0$ the term in the brackets which is proportional to $1/x^2$ dominates. The equation becomes

$$\partial_x^2 R + \frac{1}{x}\partial_x R - \frac{\nu^2}{x^2} R = 0$$

$$R = x^\alpha$$

$$R' = \alpha x^{\alpha-1}$$

$$R'' = \alpha(\alpha-1) x^{\alpha-2}$$

$$\alpha(\alpha-1) + \alpha - \nu^2 = 0$$

$$\alpha^2 = \nu^2$$

$$\Rightarrow \quad \alpha = \pm \nu$$

Thus, the two solutions to the Bessel equation behave as $(k\rho)^{\pm\nu}$ at small $k\rho$.

At large x the constant term in the brackets dominates. Let

$$R = x^{\alpha} \cdot \xi(x)$$

$$R' = \alpha x^{\alpha-1} \xi + x^{\alpha} \xi'$$

$$R'' = x^{\alpha} \xi'' + 2\alpha x^{\alpha-1} \xi' + \alpha(\alpha-1) x^{\alpha-2} \xi$$

$$\Rightarrow \quad x^{\alpha} \xi'' + \underline{2\alpha x^{\alpha-1}} \xi' + \alpha(\alpha-1) x^{\alpha-2} \xi + \alpha x^{\alpha-2} \xi +$$

$$+ \underline{x^{\alpha-1}} \xi' + \left(1 - \frac{\nu^2}{x^2}\right) x^{\alpha} \xi = 0$$

Choose $\alpha = -\frac{1}{2}$ $\Rightarrow$ *first derivative vanishes*

$$\xi'' + \Big(1 - \underbrace{\cancel{\frac{\nu^2 - 1/4}{x^2}}}_{\text{negligible at } x \to \infty}\Big) \xi = 0 \qquad (*)$$

$$\xi = \cos x, \ \sin x$$

$$\Rightarrow \quad R \to \frac{1}{\sqrt{\rho k}} \sin(\rho k) \quad \& \quad \frac{1}{\sqrt{\rho k}} \cos(\rho k)$$

The solution to the Bessel equation (103*) which is regular at the origin is called Bessel function $J_\nu(x)$; a singular solution is denoted as $N_\nu(x)$ (Neumann function).

Particular case : one can see from eq. (104*) that at $\nu = \pm 1/2$ the $1/x^2$ term in the equation disappears, and one has therefore

$$\xi = \frac{1}{\sqrt{x}}\, e^{\pm i x} \quad \rightarrow \quad \frac{1}{\sqrt{x}} \sin x \;;\; \frac{1}{\sqrt{x}} \cos x$$

Therefore, $J_{1/2}(x)$ and $N_{\frac{1}{2}}(x)$ are expressed in terms of elementary functions:

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

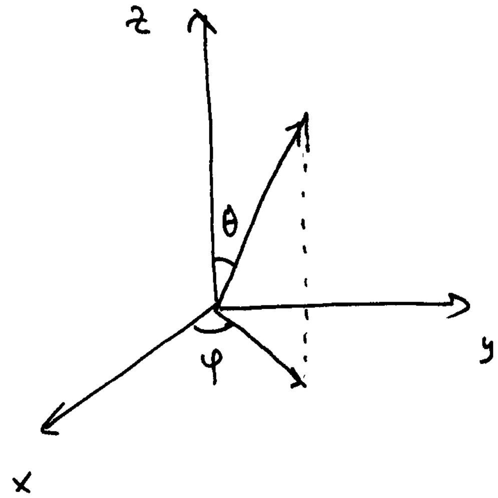
$$N_{1/2}(x) = -\sqrt{\frac{2}{\pi x}} \cos x$$

Coming back to the Laplace equation, we can write the solution in the form

$$J_n(k\rho) \cdot e^{\pm i n \varphi} \cdot e^{\pm k z}$$

Boundary conditions are satisfied by taking appropriate linear combination of these solutions.

Separation of variables in spherical coordinates



$$x = r\sin\theta\cos\varphi$$

$$y = r\sin\theta\sin\varphi$$

$$z = r\cos\theta$$

Laplace operator in coordinates r, θ, φ has the form

$$\Delta = \frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2\sin\theta}\partial_\theta \sin\theta\, \partial_\theta + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2$$

So the Laplace equation reads

$$\frac{1}{r^2}\partial_r r^2\partial_r \Phi + \frac{1}{r^2\sin\theta}\partial_\theta \sin\theta\,\partial_\theta\Phi + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2\Phi = 0 \qquad (*)$$

As in the previous cases we look for the solution in the form

$$\Phi(r,\theta,\varphi) = R(r)\cdot P(\theta)\cdot U(\phi) \qquad (**)$$

when this is substituted into the equation $(*)$ we obtain

$$\frac{1}{R}\,\frac{1}{r^2}\partial_r r^2 \partial_r R \quad + \quad \frac{1}{r^2 \sin\theta}\,\frac{1}{P(\theta)}\,\partial_\theta \sin\theta\, \partial_\theta P(\theta) \; +$$

$$+ \quad \frac{1}{U(\phi)} \cdot \frac{1}{r^2 \sin^2\theta} \cdot \partial_\varphi^2 U(\varphi) = 0$$

To separate the variables we have to require that

$$\frac{1}{U(\phi)}\,\partial_\phi^2 U(\phi) = -m^2$$

$$\Rightarrow \boxed{U(\phi) = e^{\pm i m \varphi}}$$

$$\frac{1}{\sin\theta}\,\partial_\theta \sin\theta\,\partial_\theta P + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right] P = 0 \qquad (*)$$

(Here $\ell(\ell+1)$ is just a way to represent another arbitrary constant.) Then the radial equation reads

$$\frac{1}{r^2}\,\partial_r r^2 \partial_r R - \frac{\ell(\ell+1)}{r^2} R = 0$$

Consider first the radial equation:

Writing $R = r^{\alpha}$ gives:

$$\alpha(\alpha+1) - \ell(\ell+1) = 0$$

$\Rightarrow$ two solutions for a given ℓ:

$$\alpha = \ell$$
$$\alpha = -\ell-1$$

$$(-\ell-1)\cdot(-\ell) - \ell(\ell+1) = 0. \quad \text{Ok}$$

$$\boxed{R(r) = A\,r^{\ell} + B\,r^{-\ell-1}}$$

Consider now the equation (107 *). This equation is brought to the standard form by substituting

$$x = \cos\theta \quad \Rightarrow \quad \sin^2\theta = 1-x^2$$

$$dx = -\sin\theta\, d\theta$$

$$\partial_x = \frac{d\theta}{dx}\cdot\partial_\theta \qquad \frac{d}{dx} = \frac{d\theta}{dx}\cdot\frac{d}{d\theta} = \frac{-1}{\sin\theta}\,\frac{d}{d\theta}$$

$$\frac{d}{dx}\left((1-x^2)\cdot\frac{dP}{dx}\right) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)P = 0 \qquad (*)$$

This is the <u>generalized Legendre equation</u>

At $m=0$ (particular case) one obtains the ordinary Legendre equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \ell(\ell+1)\,P = 0 \qquad (*)$$

We are interested in solutions that are finite and continuous in the interval $-1 \leqslant x \leqslant 1$. The solutions to the eq $(*)$ having such properties are called the <u>Legendre polynomials</u>. Explicitely, they have the form

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$\vdots$$

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\cdot\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

If $m \neq 0$ the solutions to eq. (108*) are the <u>associated Legendre polynomials</u> $P_e^m(x)$, $m = \pm 1, \pm 2, \dots \pm \ell$

$$P_e^m(x) = \frac{(-1)^m}{2^\ell \ell!} \cdot (1-x^2)^{m/2} \cdot \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2-1)^\ell$$

In the solution to the Laplace equation of the form (106**) it is convenient to combine the angular dependencies into a single function called <u>spherical harmonics</u> $Y_{\ell m}(\theta, \phi)$

$$Y_{\ell m}(\theta, \varphi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} \; P_\ell^m(\cos\theta)\, e^{im\varphi}$$

Few forst functions are: $\qquad Y_{\ell,-m} = (-1)^m Y^*_{\ell m}(\theta, \varphi)$

$\ell = 0$ $\qquad Y_{00} = \dfrac{1}{\sqrt{4\pi}}$

$\ell = 1$

$$\begin{cases} Y_{10} = \sqrt{\dfrac{3}{4\pi}} \cos\theta \\ Y_{11} = -\sqrt{\dfrac{3}{8\pi}} \sin\theta \cdot e^{i\varphi} \end{cases}$$

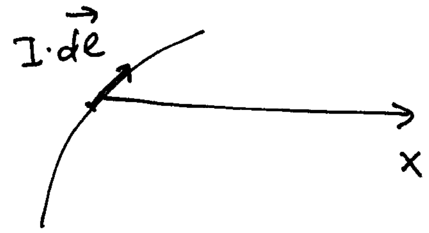
... ...

MAGNETOSTATICS

Magnetic field is described in terms of the magnetic induction $\vec{B}$ (analog of the electric field $\vec{E}$).

According to the Biot and Savart law, a current element $I \cdot d\vec{\ell}$ produces the magnetic field

$$\boxed{d\vec{B} = k \, \frac{I \, d\vec{\ell} \times \vec{x}}{|x|^3}} \qquad (*)$$



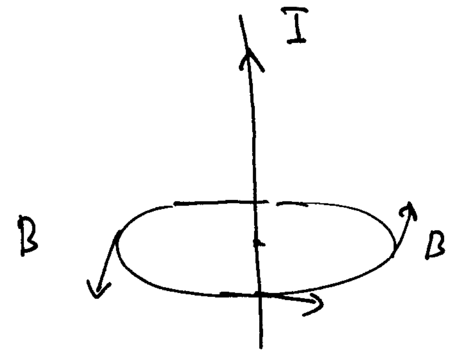
where

$$k = \frac{\mu_0}{4\pi} = 10^{-7} \cdot \frac{N}{A^2}$$

An element of force acting on the current in a magnetic field $\vec{B}$ is

$$\boxed{d\vec{F} = I \cdot d\vec{\ell} \times \vec{B}} \qquad (**)$$

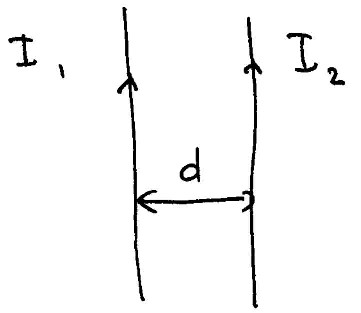
$$\vec{A} = \vec{B} \times \vec{C} \quad \Leftrightarrow \quad A_i = \varepsilon_{ijk} B_j C_k$$



Magnetic field of a linear current is

$$|\vec{B}| = \frac{\mu_0}{2\pi} \cdot \frac{I}{R}$$

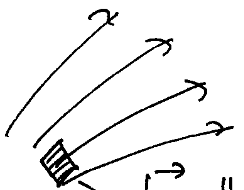
Magnetic force acting on a linear current from a parallel current:



$$\frac{dF}{dl} = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{d}$$

It is attractive if currents have the same direction.

* Current density and current conservation



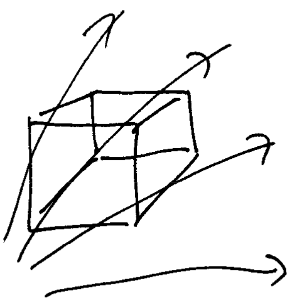
The continuous distribution of currents can be characterized by the current density $\vec{J}(\vec{x})$. By definition, the current trough a small area element $\perp$ to $\vec{J}$ is

$$dI = |\vec{J}| \cdot |ds| \quad \left(= \vec{J} \cdot d\vec{s} \text{ when not perpendicular} \right).$$

Units of $\vec{J}(\vec{x})$ are thus $\frac{A}{m^2}$.

The charge density $\rho(x)$ and the current density $\vec{J}(\vec{x})$ are related by the "<u>continuity equation</u>" which expresees conservation of electric charge.

Total charge leaving the box (per unit time):



$$\frac{\partial \vec{J}}{\partial x} \cdot dx\, \vec{n}_x \cdot dydz + \frac{\partial \vec{J}}{\partial y} \cdot dy \cdot \vec{n}_y \cdot dxdz + \frac{\partial \vec{J}}{\partial z} \cdot dz \cdot \vec{n}_z\, dxdy \simeq \vec{\nabla} \cdot \vec{J} \cdot d^3x$$

must be equal minus the change of the charge in the box, that is

$$\vec{\nabla} \cdot \vec{J}\, d^3x + \frac{\partial \rho}{\partial t} d^3x = 0 \quad \Rightarrow \quad \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0}$$

(continuity equation)

The Biot & Savart law (111*) can be written in the case of continuous current distribution $\vec{J}(\bar{x})$ as follows:

$$\vec{B}(\bar{x}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \times \underbrace{\frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}}_{-\vec{\nabla} \frac{1}{|\bar{x} - \bar{x}'|}} d^3x' \qquad (*)$$

$$= \frac{\mu_0}{4\pi} \vec{\nabla} \times \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$$

Since $\vec{B}(x)$ is a curl of a vector, its gradient is zero,

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

This is the first equation of magnetostatics.

To derive the second equation, let us calculate the curl of $(*)$. For that we will need the expression for $\vec{\nabla} \times (\vec{\nabla} \times \vec{A})$. In components,

$$\varepsilon_{ijk} \, \partial_j \underbrace{\left(\varepsilon_{kmn} \, \partial_m A_n \right)}_{\nabla \times A} =$$

$$= \underbrace{\varepsilon_{ijk}\,\varepsilon_{kmn}}_{(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})} \partial_j \partial_m A_n$$

$$= \left(\delta_{im}\,\delta_{jn} - \delta_{in}\,\delta_{jm}\right) \partial_j \partial_m A_n =$$

$$= \partial_i \left(\partial_j A_j\right) - \partial_j \partial_j A_i$$

Thus, $\vec{\nabla} \times (\vec{\nabla} \times A) = \vec{\nabla} \cdot (\vec{\nabla}\vec{A}) - \Delta \cdot \vec{A}$

and therefore

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \vec{\nabla} \int \vec{J}(x') \cdot \vec{\nabla} \frac{1}{|\bar{x} - \bar{x}'|}\, d^3x' -$$

$$- \frac{\mu_0}{4\pi} \int \vec{J}(x') \underbrace{\vec{\nabla}^2 \frac{1}{|\bar{x} - \bar{x}'|}}_{-4\pi\,\delta(x-x')} d^3x' \overset{*}{=}$$

$$\overset{*}{=} - \frac{\mu_0}{4\pi} \vec{\nabla} \int \vec{J}(x') \cdot \vec{\nabla}' \frac{1}{|\bar{x} - \bar{x}'|}\, d^3x' + \mu_0 \vec{J}(\bar{x})$$

$$= \frac{\mu_0}{4\pi} \vec{\nabla} \int d^3x' \left(\vec{\nabla}' \cdot \vec{J}(x')\right) \frac{1}{|\bar{x} - \bar{x}'|} + \mu_0 \vec{J}(\bar{x})$$

For the stationary state nothing changes with time, and the current conservation implies

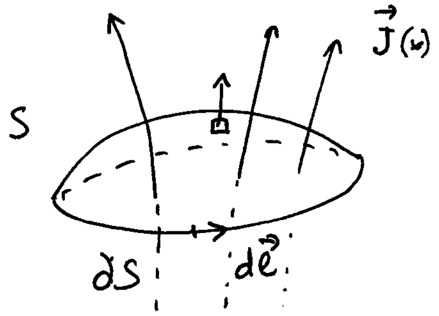
$$\cancel{\frac{\partial \rho}{\partial t}} + \vec{\nabla}\vec{J} = \vec{\nabla}\vec{J} = 0 .$$

Thus,

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}(x)} \qquad (*)$$

This is the second equation of magnetostatics.

* <u>Amere's law</u>. Integrating eq. (*) over some surface S bounded by the contour ∂S, we find:



$$\underbrace{\int_S \nabla\times B \cdot d\vec{s}} = \mu_0 \underbrace{\int \vec{J}\, d\vec{s}}_{\text{total current passing through } S}$$

$$= \int_{\partial S} \vec{B}\, d\vec{l}$$

$$\Rightarrow \boxed{\int_{\partial S} \vec{B}\, d\vec{l} = \mu_0 \int_S \vec{J}\, d\vec{s}} \qquad (**)$$

× Vector potential.

How one can solve eqs. (115*) and (115**) ?

$\bar{\nabla}\cdot\bar{B} = 0 \quad \Rightarrow$ $\vec{B}$ must be a curl of some vector field $\vec{A}(\bar{x})$ which is called the vector potential,

$$\vec{B} = \vec{\nabla} \times \vec{A}(\bar{x}) \tag{*}$$

From (113*) we have

$$\vec{A}(\bar{x}) = \frac{\mu_0}{4\pi} \cdot \int \frac{\vec{J}(x')}{|x-x'|} dx' + \vec{\nabla}\Psi(x)$$

The last term is added because the equation $\bar{\nabla}\bar{B} = 0$ does not fix $\vec{A}(\bar{x})$ completely. The vector potentials that differ by the gradient of a scalar function give the same $\vec{B}$. Adding

$$\vec{A}(x) \to \vec{A}(\bar{x}) + \vec{\nabla}\Psi$$

is called the gauge transformation. One may use this freedom to arrive at a convenient form of $\vec{A}(x)$.

Let us derive the equation for the vector potential. Substituting (116 *) into (115 *) we find:

$$\underbrace{\vec{\nabla} \times (\vec{\nabla} \times \vec{A})} = \mu_0 \vec{J}(x)$$

$$\vec{\nabla}(\bar{\nabla}\bar{A}) - \Delta \cdot \vec{A} = \mu_0 \vec{J}(x)$$

Let us now use the gauge freedom to set

$$\vec{\nabla}\vec{A} = 0.$$

[Such gauge is called <u>Coulomb gauge</u>.]

Therefore

$$\boxed{\Delta \vec{A} = -\mu_0 \vec{J}} \qquad \text{(Coulomb gauge)}$$

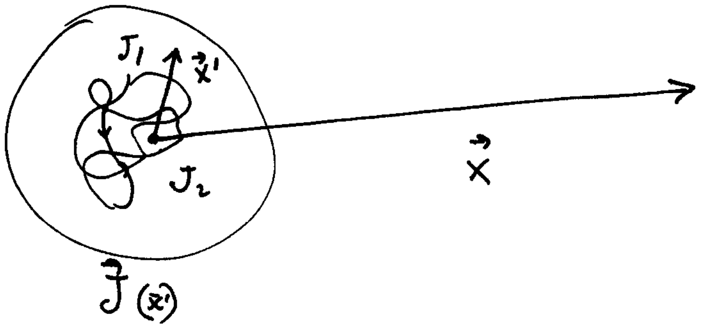
By analogy with electrostatics we can write the solution

$$\vec{A}(\bar{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(x')}{|\bar{x}-\bar{x}'|} d^3x', \qquad (*)$$

the same expression as we had before.

* Multipole expansion

Consider a current distribution which is localized in some small region of space. Let us find the magnetic induction at some large distance from this region.



We start from eq. (117 *) where we expand the denominator according to (cf. eq. 85 *):

$$\frac{1}{|x-x'|} = \frac{1}{|\vec{x}|} + \frac{\vec{x}\cdot\vec{x}'}{|x|^3} + \dots$$

Thus,

$$A_i(\vec{x}) = \frac{\mu_0}{4\pi}\left\{ \frac{1}{|x|}\int J_i(\vec{x}')\,d^3x' + \frac{\vec{x}}{|x|^3}\int J_i(\vec{x}')\cdot\vec{x}'\,d^3x' + \dots \right\} \qquad (*)$$

Consider the first contribution. We may write

$$J_i(x') = J_k(x')\,\underbrace{\partial'_k\cdot x'_i}_{\delta_{ki}}$$

Therefore,

$$\int J_i(x')\,d^3x' = \int J_k(x')\,\partial_k' x_i'\,d^3x' =$$

$$= \int d^3x' \left\{ \partial_k' \left(J_k(x')\, x_i' \right) - x_i'\, \partial_k' J_k(x') \right\}$$

First term may be transformed into the integral over the surface encircling our region. There $\vec{J} = 0$ (localized distribution!), ant the integral vanishes.
The second contribution vanishes because

$$\partial_k J_k = 0.$$

$$\Rightarrow \quad \int J_i(x')\,d^3x' = 0.$$

To rewrite the second term in eq. (118*) we use the identity

$$\partial_k \left(x_i x_j J_k \right) = x_j J_i + x_i J_j + \underbrace{x_i x_j \cdot \partial_k J_k}_{=0}$$

Integrating this equation and converting the integral in the l.h.s. into the surface integral we get

$$0 = \int dx \left(J_i(x)\, x_j + J_j(x)\, x_i \right) \qquad (*)$$

With the help of this relation we can write

$$\int d^3x' \, J_i(\bar{x}') \, x'_k = \frac{1}{2} \int d^3x' \left(J_i x'_k - J_k x'_i \right) =$$

$$= \frac{1}{2} \, \varepsilon_{ikj} \, \varepsilon_{jmn} \int d^3x' \, J_m(\bar{x}') \, x'_n$$

$$x_k \int d^3x' \, J_i(\bar{x}') \, x'_k =$$

$$= - \frac{1}{2} \, \varepsilon_{ikj} \, x_k \cdot \left\{ \int d^3x' \quad \varepsilon_{jnm} \, x'_n \, J_m \right\}$$

$$= - \frac{1}{2} \left\{ \vec{x} \times \underbrace{\int d^3x' \; \vec{x}' \times \vec{J}(\bar{x}')} \right\}_i$$

(i-th component of a double vector product)

The quantity

$$\vec{m} = \frac{1}{2} \int d^3x' \cdot \vec{x}' \times \vec{J}(\bar{x}')$$

is called the <u>magnetic moment</u> of the system of currents, while

$$\vec{\mathcal{M}}(x) = \frac{1}{2} \left[\vec{x} \times J(x) \right] .$$

is the density of the magnetic moment.

In terms of the magnetic moment $\vec{m}$, the lowest term in the expansion of the magnetic field of a localized distribution of current at large distances is

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \cdot \frac{\vec{m} \times \vec{x}}{|x|^3} \qquad \left(\propto \frac{1}{r^2} \right)$$

The magnetic field may be found by evaluating the curl of this expression:

$$\varepsilon_{ijk} \partial_j A_k = \frac{\mu_0}{4\pi} \varepsilon_{ijk} \partial_j \frac{\varepsilon_{kpq} m_p x_q}{x^3} =$$

$$= \frac{\mu_0}{4\pi} \left(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \right) m_p \, \partial_j \left(\frac{x_q}{x^3} \right) =$$

$$= \frac{\mu_0}{4\pi} \left(m_i \delta_{jq} - m_j \delta_{iq} \right) \left\{ \frac{\delta_{jq}}{x^3} - 3 \frac{x_q x_j}{x^5} \right\}$$

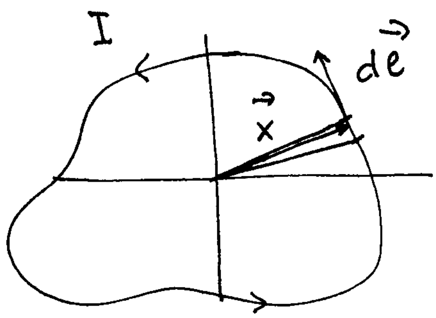
$$= \frac{\mu_0}{4\pi} \left\{ \cancel{m_i \frac{3}{x^3}} - \frac{m_i}{x^3} - \cancel{3 \frac{m_i x^2}{x^5}} + 3 \frac{(\vec{m}\vec{x}) \cdot x_i}{x^5} \right\}$$

$$\boxed{\vec{B}(x) = \frac{\mu_0}{4\pi} \frac{3\vec{x}(\vec{m}\vec{x}) - \vec{m} \cdot \vec{x}^2}{x^5}}$$

This has exactly the form of the dipole field, eq. (87*)!

The expression for the magnetic moment can be simplified in the case of a <u>flat</u> loop with the current I:

$$\vec{m} = \frac{1}{2} I \oint \vec{x} \times d\vec{l}$$



$$\vec{x} \times d\vec{l} = |\vec{x}| \cdot |dl| \cdot \sin\alpha =$$

$$= 2 \times (\underbrace{\text{area of the triangle}}_{= da}).$$

Thus,

$$\vec{m} = I \cdot \int da = I \times (\text{area of the loop}).$$

* Consider a system of charges moving inside a finite volume. Then the currents are $q_i v_i \delta^3(\bar{x} - \bar{x}_i)$.

$$\vec{J}(x) = \sum_i q_i \vec{v}_i \, \delta^3(\bar{x} - \bar{x}_i)$$

The magnetic moment of this system is, therefore,

$$\vec{m} = \frac{1}{2} \int dx' \; \vec{x}' \times \sum_i q_i \; \vec{v}_i \; \delta^3(\vec{x}' - \bar{x}_i) =$$

$$= \frac{1}{2} \sum_i q_i \cdot (\vec{x}_i \times \vec{v}_i)$$

The vector product here can be expressed in terms of the angular momentum

$$\vec{L}_i = M_i \; \vec{x}_i \times \vec{v}_i$$

M_i — mass of i-th particle.

Thus,

$$\vec{m} = \sum_i \frac{q_i}{2M_i} \vec{L}_i$$

If all particles have the same charge-to-mass ratio, the magnetic moment is expressed in terms of the <u>total</u> angular momentum,

$$\vec{m} = \frac{q}{2M} \sum_i L_i = \frac{q}{2M} \vec{L}$$

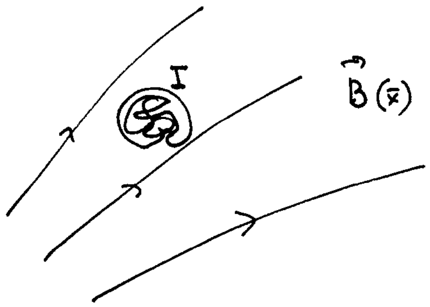
This is a well-known classical relation between angular momentum and magnetic moment, applicable, for instance, to an atom. The intrinsic moments of particles violate this relation (with $\vec{L}$ replaced by spin $\vec{S}$). For instance, the magnetic moment of electron equals (almost exactly)

$$\mu = \frac{e}{2m_e} \cdot \hbar = \mu_B \quad \text{(Bohr magneton)}$$

while the spin (intrinsic angular momentum) of electron is $\frac{1}{2}\hbar$. ($\hbar = 1{,}054 \cdot 10^{-34}$ J.s is the Planck constant)

* Force & torque acting on a localized current distribution in nearly-homogeneous magnetic field.

If the magnetic field changes slowly in space, one can replace it by its Taylor series,



$$B_k(\bar{x}) = B_k(0) + x_i \partial_i B_k(0) + \dots$$

The force acting on the current distribution $\vec{J}(\bar{x})$ is therefore

$$\vec{F} = \int d^3x \; \vec{J} \times \vec{B} \qquad (\text{cf. } 111 **)$$

or, in components,

$$F_i = \varepsilon_{ijk} \int d^3x \; J_j \left\{ B_k(0) + x_m \partial_m B_k(0) + \dots \right\}$$

First term vanishes as $\int d^3x \; \vec{J} = 0$ for any localized distribution. The second term:

$$\varepsilon_{ijk} \cdot \partial_m B_k(0) \cdot \underbrace{\int d^3x \; J_j x_m}_{} \quad \stackrel{*}{=}$$

$$\frac{1}{2} \int d^3x \left(J_j x_m - J_m x_j \right) = \qquad (\text{cf. } (119*))$$

$$= \varepsilon_{mjk} \cdot \underbrace{\varepsilon_{kpq} \cdot \frac{1}{2} \int d^3x \; x_p J_q}_{} =$$

$$= \varepsilon_{mjk} \cdot m_k$$

$$\stackrel{*}{=} \quad \varepsilon_{ijk} \, \partial_m B_k(0) \cdot \varepsilon_{mjp} \, m_p \quad =$$

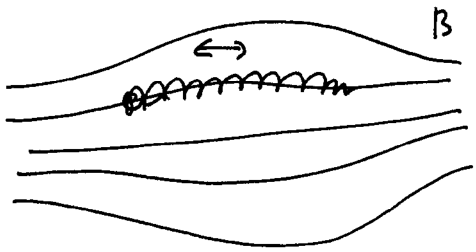
$$= \left(+ \delta_{im} \delta_{kp} - \delta_{ip} \delta_{km} \right) \partial_m B_k(0) \, m_p \; =$$

$$= \quad + \quad \partial_i \left(\vec{m} \cdot \vec{B}(0) \right)$$

Thus, we obtain

$$F = \vec{\nabla} \left(\vec{m} \cdot \vec{B} \right)$$

This force leads to the phenomenon called magnetic trapping, as will be discussed later.



Torque (moment of force) is defined in mechanics as

$$\vec{T} = \sum_i \vec{r}_i \times \vec{F}_i$$

If we have a current density $\vec{J}(x)$ in the region with non-zero magnetic field $\vec{B}(x)$, a volume d^3x is subject to the force

$$d\vec{F} = d^3x \, \vec{J} \times \vec{B}$$

Thus,
$$d\vec{T} = \vec{x} \times \left(d^3x \cdot \vec{J} \times \vec{B} \right)$$

$$\vec{T} = \int d^3x \; \vec{x} \times \left(\vec{J} \times \vec{B} \right).$$

We have therefore

$$T_i = \varepsilon_{ijk}\, \varepsilon_{kpq} \underbrace{\int d^3x \; x_j \, J_p(x) B_q(x)} \simeq$$

↳ expand $\vec{B}(x)$ and keep only first term, the constant part

$$\simeq \left(\delta_{ip}\delta_{jq} - \cancel{\delta_{iq}\delta_{jp}} \right) \cdot B_q \cdot \underbrace{\frac{1}{2} \int d^3x \left(x_j J_p - x_p J_j \right)}_{\varepsilon_{jps}\, m_s}$$

$$= \delta_{ip}\, \delta_{jq} \cdot B_q \cdot \varepsilon_{jps}\, m_s$$

$$= \varepsilon_{isq}\, m_s B_q \quad \Rightarrow \quad \boxed{\vec{T} \simeq \vec{m} \times \vec{B}}$$

Faraday induction law

Flux of $\vec{B}$: $$\Phi = \int_S \vec{B}\cdot d\vec{s}$$

"Electromotive force" $\mathcal{E}$
(line integral of $\vec{E}$) $$\mathcal{E} = \int_{\partial S} \vec{E}\, d\vec{\ell}$$

$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}}$$

Note: $d\Phi/dt$ may be due to both change of $\vec{B}$ and change of the contour ∂S.

Let us assume that the contour does not move. Then we have

$$\underbrace{\int_{\partial S} \vec{E}\, d\vec{\ell}} = -\frac{d}{dt}\int_S \vec{B}\, d\vec{s}$$

$$= \int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{s}$$

Thus,

$$\int_S d\vec{s}\left\{\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t}\right\} = 0$$

we have, in the differential form :

$$\boxed{\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0.}$$

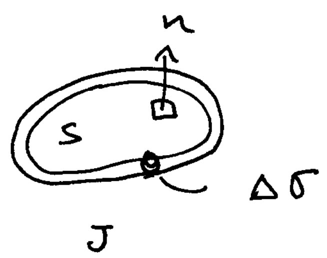
* The energy of magnetic field

In calculating the work necessary to assemble a given configuration of currents by bringing them from infinity, there is a <u>subtlety</u> related to the fact that one has to take into account the work done by the current sources needed to maintain the currents at constant values. This work is

$$\delta W = I \, \delta \Phi$$

where I is the current and $\delta\Phi$ is the change of the flux.

A distribution of currents satisfying $\vec{\nabla}\vec{J} = 0$ can be divided into contours of cross section $\Delta\sigma$. For each of these contours



$$\Delta(\delta W) = J\,\Delta\sigma \underbrace{\int_S \delta\vec{B}\cdot d\vec{s}}_{\delta\Phi} =$$

$$= J\,\Delta\sigma \int_S d\vec{s}\cdot\left(\vec{\nabla}\times\delta\vec{A}\right) =$$

$$= J\,\Delta\sigma \oint_{\partial S} \delta\vec{A}\cdot d\vec{\ell} \qquad \text{(by Stokes theorem)}$$

Since $J \Delta\sigma \cdot d\vec{\ell} = \vec{J}\, d^3x$, we obtain

$$\delta W = \int \vec{J} \cdot \delta\vec{A} \cdot d^3x$$

By making use of the Ampère's law $\frac{1}{\mu_0}\bar{\nabla}\times\bar{B} = J$ this can be rewritten as

$$\delta W = \frac{1}{\mu_0}\int \delta\vec{A} \cdot (\vec{\nabla}\times\vec{B})\, d^3x$$

Now,

$$\varepsilon_{ijk}\, \partial_i \left(\delta A_j B_k\right) = \varepsilon_{ijk}\left\{(\partial_i \delta A_j)\cdot B_k + \delta A_j (\partial_i B_k)\right\} =$$

$$= \vec{B}\cdot(\vec{\nabla}\times\delta\vec{A}) - \delta A(\bar{\nabla}\times\bar{B})$$

$$= \nabla\cdot(\delta A \times B)$$

Thus,
$$\delta W = \frac{1}{\mu_0}\int d^3x \left\{ \vec{B}\,\delta B - \cancel{\vec{\nabla}\cdot(\delta\vec{A}\times\vec{B})} \right\}$$

$$= \frac{1}{\mu_0}\,\delta\left\{\frac{1}{2}\int d^3x\, B^2\right\}$$

$$\Rightarrow \boxed{W = \frac{1}{2\mu_0}\int d^3x\, B^2}$$

Equivalently,

$$W = \frac{1}{2}\int \bar{J}\cdot\bar{A}\, d^3x \qquad (*)$$

* Coefficient of inductance

For a system of N distinct circuits one can rewrite (128 *) as

$$W = \frac{1}{2}\sum_{i=1}^{N} L_i I_i^2 + \sum_{i=1}^{N}\sum_{j>i}^{N} M_{ij} I_i I_j$$

where

$$L_i = \frac{\mu_0}{4\pi I_i^2}\int_{C_i} d^3x \int_{C_i} d^3x' \, \frac{\vec{J}(\vec{x})\cdot\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

are the coefficients of self-induction, while

$$M_{ij} = \frac{\mu_0}{4\pi I_i I_j}\int_{C_i} d^3x \int_{C_j} d^3x' \, \frac{\vec{J}(x)\cdot\vec{J}(x')}{|\bar{x}-\bar{x}'|}$$

are the coefficient of mutual inductance.

In the case of thin wires this expression reduces to

$$M_{ij} = \frac{1}{I_j}\,\Phi_{ij},$$

where Φ_{ij} is the magnetic flux created by the current j throug the circuit i (prove it!).

Maxwell's equations

Let us summarize the equations for electric and magnetic fields that we have derived so far. We will concentrate here on the local form of these equations.

1. Coulomb's law $\Rightarrow$ $$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad (*)$$

2. Biot & Savart law $\Rightarrow$ $$\vec{\nabla} \cdot \vec{B} = 0 \qquad (**)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}(x) \qquad (\vee)$$

(Ampere's law; derived by using $\vec{\nabla} \cdot \vec{J} = 0$ valid when there is no time dependence)

3. Faraday induction law

$$\Rightarrow \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 . \qquad (\vee\vee)$$

The last equation is the only time-dependent one.

In addition, there is an equation which expresses (an experimentally supported) charge conservation law:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \qquad (*)$$

First three equations (129a $*$ – v) were derived assuming static situation. There is no a priori reason why they should hold for time-dependent fields.

In fact, they cannot be correct because they are not consistent with charge conservation if the time-dependence is present. Indeed, from Ampere's law we have

$$\underbrace{\vec{\nabla} (\vec{\nabla} \times \vec{B})}_{\equiv\, 0} = \mu_0 \vec{\nabla} \cdot \vec{J} \quad \Rightarrow \quad \vec{\nabla} \vec{J} = 0$$

which contradicts to $(*)$ if $\frac{\partial \rho}{\partial t} \neq 0$.

$\Rightarrow$ <u>these equations have to be modified.</u>

The right modification was proposed by Maxwell in 1865. He noted that the current conservation equation can be converted into a vanishing divergence by means of eq. (129a*):

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = \frac{\partial}{\partial t} \cdot \varepsilon_0 \vec{\nabla} \vec{E} + \vec{\nabla} \vec{J} =$$

$$= \vec{\nabla} \left(\varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \vec{J} \right)$$

So, if one replaces $\vec{J}$

$$\vec{J} \rightarrow \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

in the Ampere's law, the consistency is recovered.

The new equation reads

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \underbrace{\varepsilon_0 \frac{\partial \vec{E}}{\partial t}}_{\text{"displacement current"}} \right)$$

The divergence of this equation gives

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B} \right) = \mu_0 \vec{\nabla} \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) =$$

$$= \mu_0 \left(\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} \right) = 0 \qquad \text{OK}.$$

The new equations predict a number of phenomena, notably the existence and properties of electromagnetic waves, which were later fully supported by experiments.

To summarize, the Maxwell's equations are:

$$\boxed{\begin{array}{ll} (1) & \vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\epsilon_0} \\ (2) & \vec{\nabla}\times\vec{B} - \mu_0\epsilon_0 \dfrac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \\ (3) & \vec{\nabla}\times\vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0 \\ (4) & \vec{\nabla}\cdot\vec{B} = 0 \end{array}}$$

Note: $\dfrac{1}{\mu_0\epsilon_0} = c^2$, where c is the speed of light.

Indeed, we have

$$\frac{1}{4\pi\epsilon_0} = 8{,}988 \cdot 10^9 \, \frac{N \cdot m^2}{C^2}$$

$$\frac{\mu_0}{4\pi} = 10^{-7} \, \frac{N}{A^2}$$

$$\Rightarrow \frac{1}{\mu_0\epsilon_0} = 8{,}988 \cdot 10^9 \cdot 10^7 \, \frac{N m^2}{C^2} \, \frac{C^2}{N s^2} = \left[3 \cdot 10^8 \, \frac{m}{s}\right]^2$$

Relativistic form of Maxwell's equations

When deriving Lorentz transformations we have seen that the equation that describes propagation of EM waves has relativistic form (we will derive this equation in the second part of the course). $\Rightarrow$ This suggests that the Maxwell's equations themselves can be rewritten in a manifestly Lorentz-invariant form, i.e., in 4-vector notations.

Let us start with charge conservation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^i} \cdot J^i = 0$$

We can write this equation as

$$\frac{\partial (c\rho)}{\partial (ct)} + \frac{\partial}{\partial x^i} J^i = \partial_\mu J^\mu = 0$$

if we introduce a 4-vector of current,

$$J^\mu = (c\rho, \vec{J})$$

Then conservation equation becomes explicitly Lorentz-invariant.

In the non-static case we have to reconsider the relations between the electric and magnetic fields $\vec{E}$ and $\vec{B}$ and the electromagnetic potentials φ and $\vec{A}$.

Since $\vec{\nabla}\cdot\vec{B} = 0$ remains valid, we have, as before

(*) $$\vec{B} = \vec{\nabla}\times\vec{A} \qquad \left(\text{equivalently, } B_i = \varepsilon_{ijk}\partial_j A_k\right)$$

However, equation $\vec{\nabla}\times\vec{E} = 0$ (which lead to $\vec{E} = -\vec{\nabla}\phi$) is no longer valid. We have instead

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

Combining this with (*) we have

$$\vec{\nabla}\times\vec{E} + \vec{\nabla}\times\frac{\partial\vec{A}}{\partial t} = \vec{\nabla}\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = 0.$$

Therefore, we obtain

$$\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi$$

or

$$\boxed{\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}} \qquad (**)$$

How to rewrite Maxwell equations in a manifestly - covariant form?

* Since ρ must be a zero-th component of a 4-vector, so must be the left hand side of eq. (129d-1). By a similar argument, the left hand side of eq. (129d-2) must transform as space components of a 4-vector.

* Since both equations contain derivatives, the fields $\vec{E}$ and $\vec{B}$ cannot transform like vectors, but must be components of some 4-dimensional tensor of the 2-nd rank.

* It follows then from eq. (129f **) that the scalar and vector potentials must be components of a 4-vector, namely

$$\mathcal{A}^\mu = (\phi, c\vec{A}) \equiv (\mathcal{A}^0, \mathcal{A}^i)$$

Then eq. (58 *) reads

$$E^i = -\frac{\partial (cA^i)}{\partial (ct)} - \frac{\partial}{\partial x^i}\phi =$$

$$= \partial^i \mathcal{A}^0 - \partial^0 \mathcal{A}^i \equiv F^{i0} \qquad (*)$$

Similarly, one can write the equation (129f*) in the form

$$B^i = -\frac{1}{2c}\,\varepsilon^{ijk}\,F^{jk}$$

where $F^{jk} = \partial^j \mathcal{A}^k - \partial^k \mathcal{A}^j$. Indeed,

$$B^i = -\frac{1}{2c}\,\varepsilon^{ijk}\left(\partial^j \mathcal{A}^k - \partial^k \mathcal{A}^j\right) =$$

$$= -\frac{1}{c}\,\varepsilon^{ijk}\,\partial^j \mathcal{A}^k = -\,\varepsilon^{ijk}\,\partial^j A^k$$

$$= \varepsilon^{ijk}\,\partial_j A^k \quad \Rightarrow \quad \vec{B} = \vec{\nabla}\times\vec{A} \qquad \text{ok}$$

We see that the electric and magnetic fields are expressed in terms of a single tensor

$$F^{\mu\nu} = \partial^\mu \mathcal{A}^\nu - \partial^\nu \mathcal{A}^\mu$$

This tensor is called the field stress tensor. It is anti-symmetric, thus there are only 6 independent components. Explicitely,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{pmatrix}$$

$$F^{ij} = -c\,\varepsilon^{ijk}\,B^k \qquad (*)$$

With these definitions of $\vec{E}$ and $\vec{B}$ in terms of the 4-vector of potential, the last two of Maxwell equations are automatically satisfied.

Let us write the first two Maxwell equations in terms of $F^{\mu\nu}$.

$$\vec{\nabla}\cdot\vec{E} = \partial_i F^{i0} = \partial_\mu F^{\mu 0} \qquad \left(\text{Note: } F^{00}=0 \text{ antisymmetry!}\right)$$

$\Rightarrow$ eq. (129d-1) becomes:

$$\boxed{\partial_\mu F^{\mu 0} = \frac{1}{c\,\varepsilon_0}\, j^0} \qquad (*)$$

$\Rightarrow$ it is covariant: both sides are 0-th component of a vector.

eq. (129d-2):

$$\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E} \longrightarrow$$

$$\varepsilon^{ijk}\,\partial_j \underbrace{\left(-\frac{1}{c}\right)\varepsilon^{kpq}\,\partial^p A^q}_{B^k} - \frac{1}{c}\,\partial_0\left(\partial^i A^0 - \partial^0 A^i\right) \overset{*}{=}$$

Product of ε - symbols:

$$\varepsilon^{ijk}\ \varepsilon^{kpq} = \varepsilon^{ijk}\ \varepsilon^{pqk} = \left(\delta^{ip}\delta^{jq} - \delta^{iq}\delta^{jp}\right)$$

$$\stackrel{*}{=} -\frac{1}{c}\left(\delta^{ip}\delta^{jq} - \delta^{iq}\delta^{jp}\right)\partial_j\,\partial^p\mathcal{A}^q - \frac{1}{c}\partial_0\left(\partial^i\mathcal{A}^0 - \partial^0\mathcal{A}^i\right) =$$

$$= -\frac{1}{c}\left(\partial^i\partial_j\mathcal{A}^j - \partial_j\partial^j\mathcal{A}^i\right) - \frac{1}{c}\partial_0\left(\partial^i\mathcal{A}^0 - \partial^0\mathcal{A}^i\right) =$$

$$= -\frac{1}{c}\partial_j\left(\partial^i\mathcal{A}^j - \partial^j\mathcal{A}^i\right) - \frac{1}{c}\partial_0\left(\partial^i\mathcal{A}^0 - \partial^0\mathcal{A}^i\right) =$$

$$= -\frac{1}{c}\partial_\mu F^{i\mu} = \frac{1}{c}\partial_\mu F^{\mu i}$$

Thus, the second equation becomes

$$\frac{1}{c}\partial_\mu F^{\mu i} = \mu_0 J^i = \mu_0 j^i$$

or, equivalently,

$$\partial_\mu F^{\mu i} = c\mu_0 j^i = \frac{c^2\mu_0}{c} j^i = \frac{1}{c\varepsilon_0} j^i$$

Combining with eq.(129i *) we finally get:

$$\boxed{\partial_\mu F^{\mu\nu} = \frac{1}{c\,\varepsilon_0}\, j^\nu}$$

Note: particlurar form of this eq. is different in different systems of units!

Electric and magnetic fields in media

So far we have discussed electric & magnetic fields in the vacuum. Sometimes this is a good approximation (e.g., in the air), but not always.

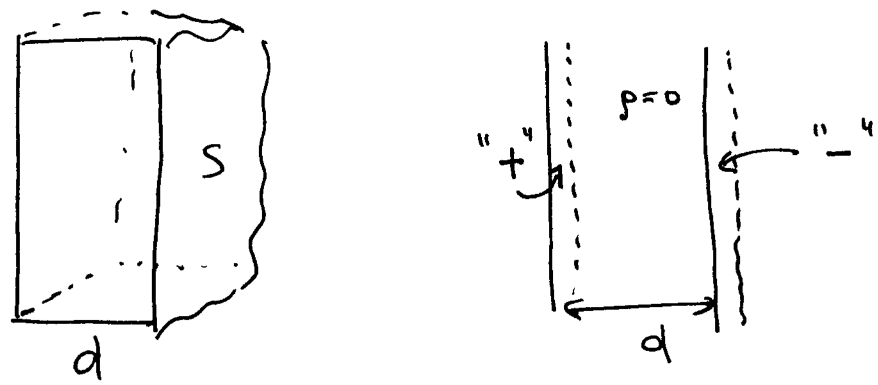
Neutral media are different from the vacuum in that they contain positive and negative charges arranged in neutral atoms. The electromagnetic properties of a given medium depend both on the properties of its atoms and on the way the atoms are combined together. Sometimes these properties differ dramatically from the vacuum ones. For instance, conductors (e.g., metals) contain charges which can move freely within the conductor. This leads to total compensation of the electric field.

The Maxwell equations which we have discussed above remain, of course, valid if we take into accoun all charges that compose a given medium, explicitely. However, this is not convenient and technically impossible. In addition, we are often not interested in such a detailed information. What we need is some sort of an averaged description valid at distances much larger than the atomic size.

There is an enormous variety of different media; these media react in a different way on external electric and magnetic fields. Here we only discuss simplest cases. They often can serve as a good approximation to real situations.

Consider a homogeneous and isotropic dielectric first. In dielectrics electrons are bound to their molecules and cannot move freely. Molecules are electrically neutral; we will also assume that they do not have an electric dipole moment (in the absence of external electric field). Thus, such a substance does not produce a macroscopic electric field.

Under the action of the external electric field positive and negative charges in a molecule are displaced in the opposite directions: in the direction of the field and in the opposite direction, respectively. Consider the effect of this displacement. Take first a thin layer of thickness d perpendicular to the displacement.



The displacement will create a surface charge density at the boundaries of our layer. This surface density can be related to the dipole moment created by the displacement:

$$\vec{P}_{tot} = Q_{tot} \cdot \vec{d} = \sigma \cdot S \cdot \vec{d} = \sigma \cdot V \cdot \vec{n}$$

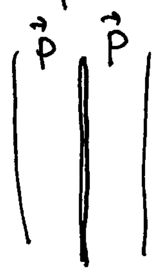
$$\Rightarrow \quad \sigma = \frac{|\vec{P}_{tot}|}{V} \equiv P = \text{dipole moment of a unit volume = polarization}$$

$\Rightarrow$ the surface charge σ is expressed in terms of the polarization $\vec{P}$ (= dipole moment of a unit volume) as

$$\sigma = |\vec{P}| \,. \tag{*}$$

What happens if we add several slices together?

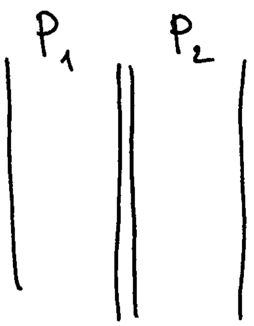
1) Equal polarizations P:



$-\sigma$ $\quad +\sigma - \sigma = 0$ $\quad +\sigma$

$\Rightarrow$ we still have $\sigma = P$

2) Different polarizations:



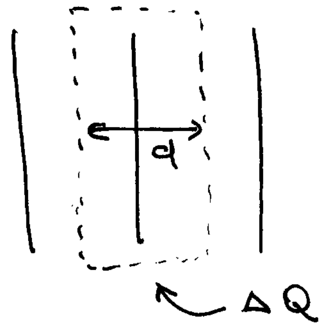
$-\sigma_1$ $\quad +\sigma_1 - \sigma_2$ $\quad +\sigma_2$

$\Rightarrow$ we get charge density in the middle!

$$\Delta Q = S \cdot (\sigma_1 - \sigma_2) = S\,(P_1 - P_2) =$$

$$= \underbrace{S \cdot d}_{V} \cdot \frac{P_1 - P_2}{d}$$

This charge density is associated with the layer of the thickness d:



In the continuum limit

$\frac{\Delta Q}{V}$ becomes $\rho_p(x)$ – charge density.

Thus, we get the relation

$$\rho_p(x) = -\frac{\partial P}{\partial x}$$

In the case of variation in all three directions this relation becomes

$$\boxed{\rho_p(x) = -\vec{\nabla}\,\vec{P}(x)}$$

At the boundaries of the dielectric, the relation (133*) is generalized as

$$\boxed{\sigma_p(x) = \vec{n} \cdot \vec{P}(x)}$$

where σ_p is the surface charge density and $\vec{n}$ is the unit normal to the surface directed out-side.

If there was an external charge distribution in addition to $\rho_p(x)$, the total charge density would be

$$\rho(x) = \rho_{ext} + \rho_p(x) =$$

$$= \rho_{ext} - \vec{\nabla}\vec{P}(x)$$

The first Maxwell equation then reads

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\left[\rho_{ext} - \vec{\nabla}\vec{P}\right]$$

This eq. can be rewritten as

$$\vec{\nabla}\left(\vec{P} + \epsilon_0 \vec{E}\right) \equiv \vec{\nabla}\vec{D} = \rho_{ext}$$

where

$$\boxed{\vec{D} = \epsilon_0 \vec{E} + \vec{P}} \qquad (*)$$

is the electric displacement.

The electric potential due to the polarization charge density $\rho_p(x)$ reads

$$\phi_p(x) = \frac{1}{4\pi\epsilon_0} \left\{ \int_V d^3x' \frac{\rho_p(x')}{|\bar{x}-\bar{x}'|} + \right.$$

$$\left. + \int_{\partial V} ds' \frac{\sigma_p(x')}{|\bar{x}-\bar{x}'|} \right\} =$$

$$= -\frac{1}{4\pi\epsilon_0} \int_V d^3x' \, \vec{P}(x') \cdot \vec{\nabla} \frac{1}{|\bar{x}-\bar{x}'|}$$

* <u>Relation between $\vec{P}$ and $\vec{E}$</u>

So far we have assumed some non-zero value of $\vec{P}$ without discussing its origin. Although the polarization may be spontaneous (in media composed of atoms with internal dipole momentum) or may be produced as a result of external conditions (for instance, applied force), the most common origin of polarization is the action of the external electric field.

The relation between the polarization vector $\vec{P}$ and the applied external electric field can be very complicated. In most cases, however, this relation is linear at small external fields,

$$P_i = \sum_j \alpha_{ij} E_j$$

If the medium is <u>isotropic</u> (i.e., does not have a preferred direction - this may not be the case in crystals), then the matrix α_{ij} is proportional to the unit matrix δ_{ij} and the directions of $\vec{P}$ and $\vec{E}$ coincide. Then the (linear) relation between them can be written as

$$\vec{P} = \epsilon_0 \chi_e \cdot \vec{E}$$

The coefficient χ_e is called the electric susceptibility of the medium. For the displacement $\vec{D}$ one may write

$$\vec{D} = \epsilon \vec{E} \qquad (*)$$

where

$$\epsilon = \epsilon_0 (1 + \chi_e)$$

is the electric permittivity. The ratio $\frac{\epsilon}{\epsilon_0} = 1 + \chi_e$ is called the dielectric constant or relative electric permittivity.

For isotropic media the divergence equation can be written as

$$\vec{\nabla} \vec{E} = \frac{\rho}{\epsilon}$$

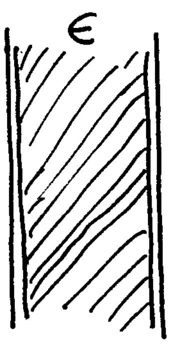
It is clear, therefore, that the solutions for electric field in an isotropic medium can be obtained from corresponding solutions in the vacuum by replacing $\epsilon_0 \to \epsilon$.

For instance, the electric field of a charge q embedded in a dielectric with the permittivity ϵ is (assuming the dielectric is infinite)

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon} \cdot q \cdot \frac{\vec{x}}{|x|^3}$$

Note that since $\epsilon > \epsilon_0$, this electric field is weaker than it would be in the vacuum. This is obvious on physical grounds: the charges in the dielectric move in such a way as to compensate the external field. The external charge is screened (partially) by polarization of the medium.

Likewise in a capacitor filled with a dielectric the electric field is weaker, than it would be in the vacuum for the same charge,



$$E = \frac{\sigma}{\epsilon}$$

Thus, the capacity becomes larger,

$$C = \frac{Q}{V} = \frac{\epsilon}{\epsilon_0} C_{vac}$$

To fully specify the problem of finding the electric field for a given charge distribution in presence of dielectrics, we need also to determine the boundary conditions at the boundary between two different media. These conditions follow from the Maxwell equations, which in our case are

$$\vec{\nabla}\vec{D} = \rho_{ext}$$

$$\vec{\nabla}\times\vec{E} = 0$$

<u>external</u> charge density <u>not including</u> polarization charge!

The second of these equations gives continuity of the tangential component of $\vec{E}$, which can be written as

$$\boxed{\vec{n}_{21}\times\left(\vec{E}_2-\vec{E}_1\right) = 0}$$

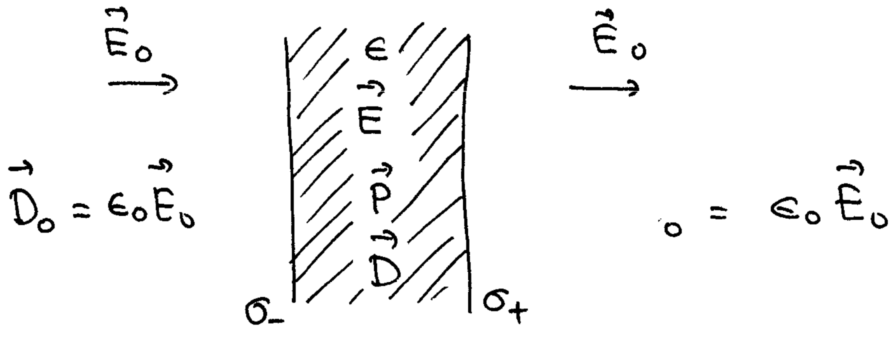
unit normal vector directed from 1 to 2

The first equation implies

$$\boxed{\vec{n}_{21}\cdot\left(\vec{D}_2-\vec{D}_1\right) = \sigma_{ext}}$$

surface density of external charge

Consider an example: slice of dielectric in a uniform electric field $\vec{E}_0$



Two ways to deal with the problem

"<u>macroscopic</u>"

$$\left.\begin{array}{l}\vec{\nabla}\cdot\vec{D} = 0 \\ \vec{n}\cdot(\vec{D}_2 - \vec{D}_1) = 0\end{array}\right\} \longrightarrow \vec{D} = \vec{D}_0$$

(no external charges)

$\Rightarrow$ $\vec{D}$ is the same inside and outside.

Then from $\vec{D} = \epsilon \vec{E}$

$$\vec{E} = \frac{\epsilon_0}{\epsilon}\vec{E}_0$$

$$\vec{P} = \epsilon_0 \chi \vec{E} = (\epsilon - \epsilon_0)\vec{E}$$

$$= \frac{\epsilon_0}{\epsilon}(\epsilon - \epsilon_0)\vec{E}_0$$

"<u>microscopic</u>"

surface charge densities $\pm\sigma$ create an extra field $\frac{\sigma}{\epsilon_0}$:

$$E = E_0 - \frac{\sigma}{\epsilon_0} = E_0 - \frac{P}{\epsilon_0}$$

(note: $P = \sigma$)

Polarization is proportional to $\vec{E}$,

$$P = \epsilon_0 \chi E = (\epsilon - \epsilon_0)E$$

Now solve these two eqs for P and E:

$$E = \frac{\epsilon_0}{\epsilon} E_0$$

$$P = (\epsilon - \epsilon_0)\frac{\epsilon_0}{\epsilon} E_0$$

OK, coincides.

* Energy

To derive the expression for the energy of a given charge distribution, we cannot use directly the rule "$\epsilon_0 \to \epsilon$". This is because it has been derived by the assembly of charges brought from infinity, and now we also have to account for the work spent to polarize the medium. So, we start anew from the relation

$$\delta W = \int d^3x \;\; \delta\rho_{ext}(x)\, \Phi(x)$$

external charge density

potential, including the polarizations effects

From the equation $\vec{\nabla}\vec{D} = \rho_{ext}$ we can replace $\delta\rho$ by $\vec{\nabla}\cdot\delta\vec{D}$,

$$\delta W = \int d^3x \;\; \vec{\nabla}\,\delta\vec{D}\cdot\Phi$$

(by parts setting boundary terms to 0)

$$= \int d^3x\, \delta\vec{D}\cdot\left(-\vec{\nabla}\Phi\right) =$$

$$= \int d^3x \;\; \delta\vec{D}\cdot\vec{E}$$

This expression is valid for any relation between $\vec{P}$ and $\vec{E}$, linear or not. It implies

$$W = \int d^3x \int_0^D \vec{E} \cdot \delta\vec{D}$$

If the relation between $\vec{E}$ and $\vec{D}$ is linear as in eq. (138*), the last integral can be evaluated to, $\frac{1}{2}\vec{E}\cdot\vec{D}$. Thus, in this case we have

$$\boxed{W = \frac{1}{2}\int d^3x \; \vec{E}\cdot\vec{D}}$$

If we use the relation $\vec{E} = -\vec{\nabla}\phi$ we can rewrite this equation in the form

$$W = \frac{1}{2}\int d^3x \; \rho(x)\,\Phi(x),$$

formally analogous to the vacuum equation.

* Energy of a dielectric in an external field

Let $\vec{E}_0$ be an external field in a medium of the electrical susceptibility ϵ_0. Its electrostatic energy is therefore

$$W_0 = \frac{1}{2}\int \bar{E}_0 \bar{D}_0 \, d^3x ,$$

where $\vec{D}_0 = \epsilon_0 \vec{E}_0$

Suppose a dielectric object of volume V and susceptibility ϵ_1 is introduced into the field changing it from $\vec{E}_0$ to $\vec{E}$. Let $\epsilon(x)$ be a susceptibility that equals ϵ_0 outside V and ϵ_1 inside V. The energy of the system changes therefore to

$$W = \frac{1}{2}\int \vec{E}\cdot\vec{D}$$

where $\vec{D} = \epsilon\vec{E}$. The difference in energy is

$$\Delta W = W - W_0 = \frac{1}{2}\int d^3x \, (\bar{E}\cdot\bar{D} - \bar{E}_0 \bar{D}_0)$$

$$= \frac{1}{2}\int d^3x \, (\bar{E}\cdot\bar{D}_0 - \bar{D}\cdot\bar{E}_0) \quad +$$

$$+ \underbrace{\frac{1}{2}\int d^3x \, (\bar{E}+\bar{E}_0)(\bar{D}-\bar{D}_0)}_{I_2}$$

The second integral is zero. Indeed, since $\nabla\times(E+E_0) = 0$ one may write $E + E_0 = -\nabla\phi$

$$I_2 = -\frac{1}{2}\int d^3x \, \vec{\nabla}\phi\,(\vec{D}-\vec{D}_0) =$$

$$= \frac{1}{2}\int d^3x \, \phi\,\vec{\nabla}(\vec{D}-\vec{D}_0) = 0 \quad \left(\begin{array}{l}\text{external charges are}\\ \text{the same for } D \text{ and } D_0\end{array}\right)$$

Thus,

$$\Delta W = \frac{1}{2} \int d^3x \left(\vec{E} \cdot \vec{D}_0 - \vec{D} \cdot \vec{E}_0 \right) =$$

$$= - \frac{1}{2} \int d^3x \left(\epsilon - \epsilon_0 \right) \vec{E} \cdot \vec{E}_0$$

$$= - \frac{1}{2} \int_V d^3x \left(\epsilon_1 - \epsilon_0 \right) \vec{E} \cdot \vec{E}_0$$

The change in energy is negative for $\epsilon_1 > \epsilon_0$. Thus, such objects are <u>attracted</u> into regions of stronger field.

Note that the expression for energy can be written in terms of polarization P in the case when the first medium is vacuum. Since according to (136*) $P = (\epsilon - \epsilon_0) E$,

$$\Delta W = - \frac{1}{2} \int_V d^3x \; \vec{P} \cdot \vec{E}_0$$

which is the energy of a dipole except for the factor $1/2$.

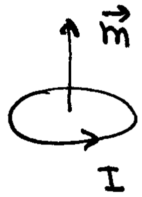
Magnetostatics in media

The logic which we have followed in the case of electric field and polarization is applicable in the case of magnetic field as well. At zero external magnetic field many substances have no macroscopic currents (magnetic fields) associated with them. At the microscopic level, however, such currents are always present. The motion of electrons produces currents which may give rise to magnetic moments. In addition, electrons and nuclei have intrinsic magnetic moments which contribute to the microscopic magnetic field. At zero external magnetic field these microscopic fields average to zero for many substances (but not for all!).

Even if in the absence of the magnetic field the magnetic moments of atoms were absent, they are induced when the field is applied, just like the electric dipoles are induced by the external electric field.

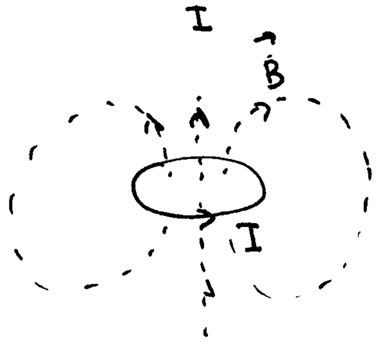
<u>Reminder</u>:

$$\vec{m} = \frac{1}{2}\int d^3x \;\; \vec{x} \times \vec{J}(x)$$



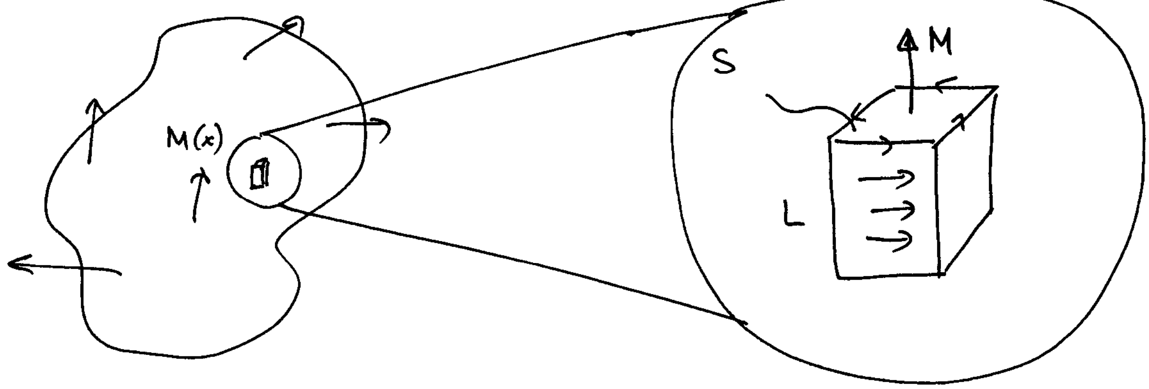
$|m| = I \cdot A$

$\quad$ ↳ area of the loop



In the external magnetic field a magnetic moment orients itself in such a way that it becomes $\parallel$ to $\vec{B}$.

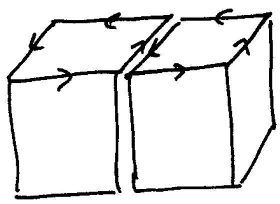
Consider a magnetized body (i.e., a body in which microscopic magnetic moments do not average to zero). Consider a small region such that it can be considered as homogeneous



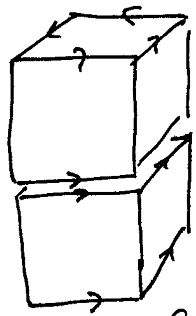
Magnetic moment of this piece is created by the surface current with current density j, so that $j \cdot L = I$ is the total surface current.
We have

$$M_{tot} = S \cdot I = S \cdot L \cdot j = V \cdot j \qquad (*)$$

If we put two such blocks together, the magnetic moment doubles:



area doubles



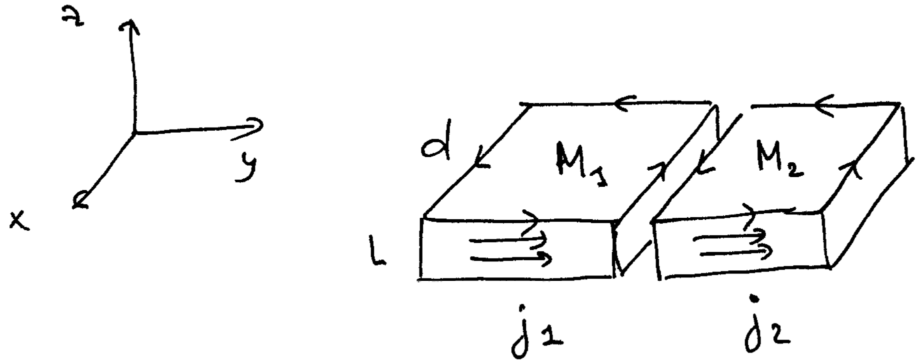
length doubles.

$\Rightarrow$ Our body at a given point can be characterized by the <u>magnetic moment per unit volume</u> $\vec{M}(x)$, which is a <u>local</u> quantity. From $(*)$ we have

$$M(x) = \frac{M_{tot}}{V} = j$$

This is magnetic analog of the relation $P = \sigma$.

Now consider what happens when magnetization $\vec{M}(x)$ varies with space. Take two blocks with different magnetization and put them together:



$M_1 = j_1$

$M_2 = j_2$

Two currents in the middle do not compensate each other. The net current is

$$\Delta J = L \cdot (j_2 - j_1) =$$
$$= L \cdot d \, \frac{j_2 - j_1}{d} =$$
$$= L \cdot d \, \frac{M_2 - M_1}{d}$$

$$\frac{\Delta J}{L \cdot d} = \frac{M_2 - M_1}{d} \quad \underset{\text{continuum limit}}{\Longrightarrow} \quad J_x = \frac{\partial M_z}{\partial y}$$

This is actually a part of the relation

$$\boxed{\vec{J}_M(x) = \vec{\nabla} \times \vec{M}(x)} \qquad (*)$$

↳ magnetization current

Let us rederive eq. (148*) in a different way. The vector potential, created by an elementary magnetic dipole $\vec{M}(x) \cdot \Delta V$ is

$$\Delta \vec{A}(x) = \frac{\mu_0}{4\pi} \cdot \frac{\vec{M} \times (\bar{x} - \bar{x}')}{|\bar{x} - \bar{x}'|^3} \Delta V$$

(cf. eq. 121*). Thus, the magnetized medium and the external current together create the potential

$$A(x) = \frac{\mu_0}{4\pi} \int d^3x' \left\{ \frac{\vec{J}_{ext}(x')}{|\bar{x} - \bar{x}'|} + \frac{\vec{M}(x') \times (x - x')}{|\bar{x} - \bar{x}'|^3} \right\}$$

The last term we can write as

$$\int \frac{M(x') \times (\bar{x} - \bar{x}')}{|\vec{x} - \vec{x}'|^3} d^3x' = \int \vec{M}(x') \times \vec{\nabla}' \frac{1}{|\vec{x} - \vec{x}'|} d^3x'$$

$$= \int \vec{\nabla}' \times \vec{M}(\bar{x}') \cdot \frac{1}{|\bar{x} - \bar{x}'|} d^3x'.$$

Thus,

$$\vec{A}(x) = \frac{\mu_0}{4\pi} \int \frac{\left(\vec{J}_{ext}(x') + \vec{\nabla}' \times M(\bar{x}')\right)}{|x - \vec{x}'|} d^3x'$$

Comparing to the standard expression for the vector potential we see that the external current is replaced by the combination

$$\vec{J}_{ext}(x) + \vec{\nabla} \times \vec{M}(x)$$

Thus, we arrive at eq. (148*).

* The total macroscopic magnetic field is determined by the sum of the effective current due to magnetization and an external current $\vec{J}_{ext}(x)$ by the Maxwell equation:

$$\vec{\nabla} \times \vec{B} = \mu_0 \left[\vec{J}_{ext} + \vec{\nabla} \times \vec{M} \right]$$

This can be rewritten as

$$\vec{\nabla} \times \vec{H} = \vec{J}_{ext}$$

where

"magnetic field" "magnetic induction"

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$$

* Relation between $\vec{B}$ and $\vec{H}$.

In order to obtain closed system of equations, we need to supplement it with the relation between $\vec{B}$ and $\vec{H}$. This is where all complications related to the material structure come into play.

For isotropic <u>diamagnetic</u> and <u>paramagnetic</u> substances this relation is linear

$$\vec{B} = \mu \vec{H}$$

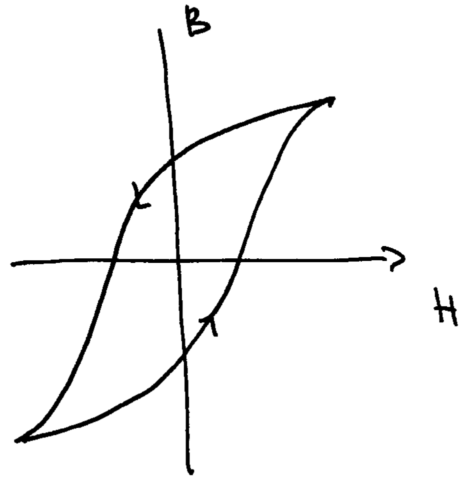
μ ← magnetic permeability

$\mu > \mu_0$ for paramagnetics
$\mu < \mu_0$ for diamagnetics
typically $\mu/\mu_0 - 1 \sim 10^{-5}$

For the <u>ferromagnetic</u> substances this relation is non-linear,

$$\vec{B} = \vec{F}(\vec{H})$$

Moreover, the phenomenon of hysteresis takes place, i.e. the magnetic induction $\vec{B}$ is not a single-valued function of H, but depends on the history.



The permeability $\mu(H)$ is defined as $\frac{dB}{dH}$. The value of $\mu(H)/\mu_0$ can be as high as 10^6

* Boundary conditions at the boundary between regions of different magnetic permeability can be derived in a standard way from the equations

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_{ext}$$

These equations give

$$(\vec{B}_2 - \vec{B}_1)\,\vec{n}_{21} = 0$$

$$\vec{n}_{21} \times (\vec{H}_2 - \vec{H}_1) = j_{ext}$$

where $\vec{n}_{21}$ is the normal to the surface pointing from region 1 into region 2, while j_{ext} is the surface current density.

<u>* Energy</u>. The method which we used to derive the expression for the energy of the magnetic field is applicable in the presence of permeable medium as well. In particular, the equation

$$\delta W = \int \delta\vec{A}\cdot\vec{J}_{ext}\, d^3x \qquad (*)$$

is still valid in this case. In the same way as we have done it in the vacuum, it can be converted with the help of the equation

$$\vec{\nabla}\times\vec{H} = \vec{J}_{ext}$$

Then one uses the identity

$$\vec{\nabla}\cdot(\vec{P}\times\vec{Q}) = Q\cdot(\vec{\nabla}\times\vec{P}) - \vec{P}\cdot(\vec{\nabla}\times\vec{Q})$$

to convert eq. (152*) into

$$\delta W = \int \Big(\vec{H}\cdot\underbrace{\vec{\nabla}\times\delta\vec{A}}_{\delta\vec{B}} + \underbrace{\cancel{\nabla\cdot(H\times\delta A)}}_{=0}\Big)\, d^3x$$

$=0$ for localized distribution of fields

Thus,

$$\boxed{\delta W = \int \vec{H}\cdot\delta\vec{B}\; d^3x}$$

If the relation between $\vec{H}$ and $\vec{B}$ is linear, this equation gives

$$\boxed{W = \frac{1}{2}\int \vec{H}\cdot\vec{B}\; d^3x}$$

(for linear relation between $\vec{H}$ and $\vec{B}$)

Maxwell's eqs in medium:

To summarize, in the static case we have:

$$\vec{\nabla}\cdot\vec{D} = \rho_{ext}$$

$$\vec{\nabla}\times\vec{E} = 0 \qquad (*)$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{H} = \vec{J}_{ext}$$

plus two eqs. which relate $\vec{D}$ and E and $\vec{B}$ and $\vec{H}$.
In the simplest linear case these equations are

$$\vec{D} = \varepsilon\vec{E}$$

$$\vec{B} = \mu\vec{H}$$

In the case of time-dependent fields and densities the Faraday induction law is still valid, so the second equation has to be replaced by

$$\vec{\nabla}\times\vec{E} = 0 \quad\rightarrow\quad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

As in the vacuum, the last equation is not compatible with conservation of external charges, which can be written in the form

$$\frac{\partial\rho_{ext}}{\partial t} + \vec{\nabla}\vec{J}_{ext} = 0 \quad\Rightarrow\quad \vec{\nabla}\left(\frac{\partial}{\partial t}\vec{D} + \vec{J}_{ext}\right) = 0 .$$

Replacing the external current in the last eq (153a*) by this combination with zero divergence, we get the equation

$$\vec{\nabla} \times \vec{H} = \vec{J}_{ext} \quad \longrightarrow \quad \vec{\nabla} \times \vec{H} - \frac{\partial D}{\partial t} = \vec{J}_{ext}.$$

The resulting set of equations are Maxwell's equations in the medium

$$\boxed{\begin{aligned} \vec{\nabla}\vec{D} &= \rho_{ext} \\ \vec{\nabla} \times \vec{H} - \frac{\partial D}{\partial t} &= \vec{J}_{ext} \\ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0 \end{aligned}}$$

They have to be supplemented by the relation between $\vec{B}$ and $\vec{H}$ and between $\vec{D}$ and $\vec{E}$, which in the linear case reads

$$\vec{D} = \varepsilon \vec{E}$$
$$\vec{B} = \mu \vec{H}$$

In the vacuum $\varepsilon = \varepsilon_0$, $\mu = \mu_0$ and one recovers the Maxwell's equations in vacuum.

CHARGED PARTICLE IN ELECTROMAGNETIC FIELD

* Equation of motion

The non-relativistic equation of motion of a charged particle in the electric field $\vec{E}$ and magnetic field $\vec{B}$ is given by the Newton's law,

$$m \frac{d\vec{v}}{dt} = e \left\{ \vec{E} + \vec{v} \times \vec{B} \right\}$$

particle mass particle charge

As we have discussed in the first part of these lectures, the relativistic generalization of this equation is obtained by replacing $m \frac{d\vec{v}}{dt}$ with $\frac{d\vec{p}}{dt}$ where

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}$$

is the relativistic momentum. Thus, the relativistic equation reads

$$\boxed{\frac{d\vec{p}}{dt} = e\left\{\vec{E} + \vec{v}\times\vec{B}\right\}} \qquad (*)$$

This equation does not have a relativistic form. To see that it is a relativistic equation, let us rewrite it in terms of 4-vectors.

As we have already discussed, the relativistic equation of mechanics has the form

$$mc\,\frac{du^\mu}{ds} = f^\mu \qquad ; \quad u^\mu = \frac{dx^\mu}{ds}$$

where

$$f^\mu = \left(\frac{(\vec{F}\cdot\vec{v})}{c\sqrt{1-v^2/c^2}} \,,\; \frac{\vec{F}}{\sqrt{1-v^2/c^2}} \right)$$

is the 4-force (it has 3 independent components as it is subject to the constraint $u_\mu f^\mu = 0$), $ds = \sqrt{dx_\mu dx^\mu}$ is the (infinitesimal) interval.

Thus, the l.h.s. of eq. $(*)$ is the space part of the expression

$$\frac{dp^\mu}{ds} = m\,\frac{du^\mu}{ds} = \frac{m}{c\sqrt{1-v^2/c^2}}\cdot\frac{du^\mu}{dt}$$

How to rewrite

$$\frac{1}{\sqrt{1-v^2/c^2}}\; e\,\left\{\vec{E} + \vec{v}\times\vec{B}\right\} \qquad (*)$$

in the form of 4-vectors? (It must be a space component of some vector to match the l.h.s).

This combination is linear in electric and magnetic fields. The latter, as we know, form a 4-tensor (cf. eqs. 60*) and (61*)) $\hat{F}^{\mu\nu} = -\hat{F}^{\nu\mu}$:

$$\hat{F}^{i0} = E_i \qquad \left(= \hat{\partial}^i\hat{A}^0 - \hat{\partial}^0\hat{A}^i\right)$$

$$\hat{F}^{ij} = -c\,\varepsilon_{ijk}B_k \qquad \left(= \partial^i A^j - \partial^j A^i\right)$$

A comment on indices: in this calculation we have both 3-dimensional and 4-dimensional vectors (tensors). To avoid confusion, we mark 4-vectors with hat. Thus,

$$\hat{u}^\mu = \frac{1}{\sqrt{1-v^2/c^2}}\,(c,\, v_i) \quad \Rightarrow \quad \hat{u}^i = \frac{v_i}{\sqrt{}}$$

$$\hat{u}_\mu = \frac{1}{\sqrt{1-v^2/c^2}}\,(c,\, -v_i) \qquad \hat{u}_i = \frac{-v_i}{\sqrt{}}$$

v_i – components of 3-velocity

The expression (156*) also contains velocity. The only covariant combination that one can make out of $\hat{F}^{\mu\nu}$ and $\hat{u}^\mu$ is $\hat{F}^{\mu\nu}\hat{u}_\nu$. Let us calculate its spatial component,

$$\hat{F}^{i\mu}\hat{u}_\mu = \hat{F}^{i0}u_0 + \hat{F}^{ij}\hat{u}_j =$$

$$= \frac{c\,E_i}{\sqrt{1-v^2/c^2}} - c\,\varepsilon_{ijk}B_k \cdot \frac{-v_j}{\sqrt{1-v^2/c^2}} =$$

$$= \frac{c}{\sqrt{1-v^2/c^2}}\left(E_i + \varepsilon_{ijk}v_j B_k\right) =$$

$$= \frac{c}{\sqrt{1-v^2/c^2}}\left\{\vec{E} + \vec{v}\times\vec{B}\right\}_i$$

Thus, we see that eq. (155*) is precisely the space component of the covariant equation

$$\boxed{\frac{d\hat{p}^\mu}{ds} = \frac{e}{c^2}\hat{F}^{\mu\nu}\cdot\hat{u}_\nu} \qquad (*)$$

l.h.s: $\dfrac{d\hat{p}^i}{ds} = \dfrac{1}{c\sqrt{1-v^2/c^2}} \cdot \dfrac{d\hat{p}^i}{dt}$

r.h.s: $\dfrac{e}{c^2}\hat{F}^{i\nu}u_\nu = \dfrac{e}{c\sqrt{1-v^2/c^2}}\left\{\vec{E} + \vec{v}\times\vec{B}\right\}_i$

Equating the two one recovers eq. (155*).

$\Rightarrow$ we are dealing indeed with relativistic equation

The time component of eq (157*) is, in fact, the equation of energy conservation relating the change in particle energy to the work done by electric field - check it!

This equation has the form

$$\boxed{\frac{d\mathcal{E}}{dt} = e\,\vec{v}\cdot\vec{E}} \qquad (*)$$

Note that the magnetic field produces force which is perpendicular to the velocity, and thus this force does not produce any work.

* Motion in a uniform electric field.

Let us choose the coordinate system in such a way that x-axis coincides with the direction of the electric field, while the initial velocity of the particle lies in the plane (xy). The motion will be 2-dimensional in the plane (xy).

We have

$$\frac{dp_x}{dt} = eE$$

$$\frac{dp_y}{dt} = 0.$$

Thus,

$$\left\| \begin{array}{l} p_x = eEt \\ p_y = p_0 = \text{const.} \end{array} \right.$$

(we have chosen the time in such a way that $p_x = 0$ at $t = 0$)

The kinetic energy of the particle is

$$\mathcal{E} = c\sqrt{m^2c^2 + p^2}$$

$$= \sqrt{m^2c^4 + c^2p_0^2 + (ceEt)^2} =$$

$$= \sqrt{\mathcal{E}_0^{\,2} + (ceEt)^2}$$

$$\mathcal{E}_0 = \sqrt{m^2c^4 + c^2p_0^2}$$

The kinetic energy is related with velocity and momentum by (43*).

$$\vec{p} = \frac{\vec{v}\,\mathcal{E}}{c^2} \qquad (*)$$

Therefore,

$$v_x = \frac{dx}{dt} = \frac{c^2}{\mathcal{E}}\,p_x = \frac{c^2 eEt}{\sqrt{\mathcal{E}_0^2 + (ceEt)^2}}$$

Integrating this relation we get

$$\boxed{x(t) = x_0 + \frac{1}{eE}\sqrt{\mathcal{E}_0^2 + (ceEt)^2}}$$

The same eq. (160*) may be used to determine $y(t)$. We have

$$\frac{dy}{dt} = \frac{p_0 c^2}{\mathcal{E}} = \frac{p_0 c^2}{\sqrt{\mathcal{E}_0^2 + (ceEt)^2}}$$

Integrating this equation we find

$$y(t) = y_0 + \frac{p_0 c}{eE} \operatorname{arcsh} \frac{ceEt}{\mathcal{E}_0} \qquad (*)$$

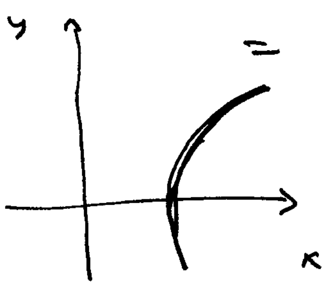
To find the trajectory one may use eq (*) to express t as a function of y,

$$t = \frac{\mathcal{E}_0}{ceE} \operatorname{sh}\left\{ \frac{Ee}{cp_0}(y-y_0) \right\}$$

$$ceEt = \mathcal{E}_0 \cdot \operatorname{sh}\{\ldots\}$$

$$x = x_0 + \frac{1}{eE}\sqrt{\mathcal{E}_0^2 + \mathcal{E}_0^2 \operatorname{sh}^2\{\ldots\}} =$$

$$= x_0 + \frac{\mathcal{E}_0}{eE} \operatorname{ch}\left\{ \frac{Ee}{cp_0}(y-y_0) \right\}$$



* Motion in a uniform magnetic field.

In the case of a uniform, static magnetic field the equations (155*) and (158*) become

$$\begin{cases} \dfrac{d\vec{p}}{dt} = e\,\vec{v}\times\vec{B} \\[2ex] \dfrac{d\mathcal{E}}{dt} = 0 \end{cases}$$

The second equation means that the kinetic energy of the particle is constant, and therefore its velocity has constant value (but not the direction!)

$$|\vec{v}| = \text{const} = v$$

Then

$$\frac{d}{dt}\frac{m\vec{v}}{\sqrt{1-\vec{v}^2/c^2}} = \frac{m}{\sqrt{1-\vec{v}^2/c^2}}\cdot\frac{d\vec{v}}{dt} \equiv m\gamma\frac{d\vec{v}}{dt}$$

$$\gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}}$$

$\Rightarrow$ we get the equation

$$m\gamma\frac{d\vec{v}}{dt} = e\,\vec{v}\times\vec{B}$$

or

$$\frac{d\vec{v}}{dt} = \vec{v} \times \vec{\omega}_B \qquad (*)$$

Here

$$\vec{\omega}_B = \frac{e\vec{B}}{m\gamma} = \frac{ec^2\vec{B}}{\mathcal{E}}$$

is called the <u>gyration</u> frequency.

To solve this equation choose the system of coordinates with the axis z in the direction of $\vec{\omega}_B$, so that

$$\vec{\omega}_B = (0, 0, \omega_B).$$

Then the z-component of equation $(*)$ reads

$$\frac{dv_z}{dt} = 0 \quad \Rightarrow \quad \boxed{v_z = \text{const.}}$$

The other two equations are

$$\frac{dv_x}{dt} = v_y \cdot \omega_B$$

$$\frac{dv_y}{dt} = -v_x \cdot \omega_B$$

$$\frac{d}{dt}\left(v_x + i v_y\right) = -i\,\omega_B\left(v_x + i v_y\right)$$

Thus,

$$v_x + i v_y = a\, e^{-i\omega t}$$

↳ write as $v_0 e^{-i\alpha}$

$$v_x + i v_y = v_0\, e^{-i(\omega t + \alpha)}$$

$$\Rightarrow \quad \begin{aligned} v_x &= v_0 \cos(\omega t + \alpha) \\ v_y &= -v_0 \sin(\omega t + \alpha) \end{aligned} \qquad (*)$$

Obviously, the constants v_z and v_0 satisfy the equation

$$v^2 = v_z^2 + v_0^2$$

Integrating equations $(*)$ we get

$$x(t) = x_0 + R \sin(\omega t + \alpha)$$

$$y(t) = y_0 + R \cos(\omega t + \alpha)$$

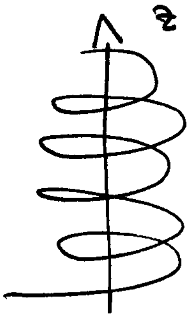
where the constant R is

$$R = \frac{v_0}{\omega_B} = \frac{v_0 \mathcal{E}}{ec^2 B} = \frac{p_\perp}{eB} \qquad (*)$$

momentum perpendicular to $\vec{B}$

For the z component we have

$$z = z_0 + v_z t$$



The motion therefore is helical with the axis directed along the magnetic field and the radius R given by eq. $(*)$.

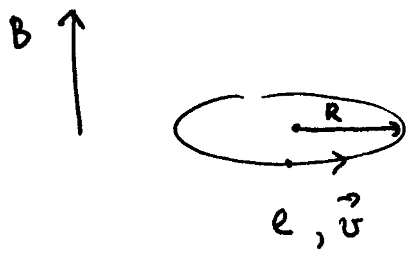
<u>Note</u>: the radius of particle trajectory depends on particle energy. This is used to measure energies of charged particles.

In the non-relativistic limit we have $\mathcal{E} \sim mc^2$ and eq $(*)$ becomes

$$R \sim \frac{v_0 m}{e \cdot B}$$

* Motion in a slowly-varying magnetic field.

In many situations the magnetic field does not substantially change at distances of order of the radius of particle orbit R, or at times of order of inverse frequency of the circular motion ω_B^{-1}. In these situations one may work out a very useful approximation based on the existance of adiabatic invariants (see course of analytical mechanics).



For a particle moving in the magnetic field along the circular orbit of radius R one can define a current

$$I = \frac{e}{T} = \frac{1}{2\pi} e\omega_B$$

period of the circular motion

One can associate a magnetic moment to such a particle,

$$m = I \times (\text{area}) = I \cdot \pi R^2$$

$$= \frac{1}{2} e \omega_B R^2$$

Making use of the relations

$$\omega_B = \frac{e c^2 B}{\mathcal{E}} \qquad R = \frac{v_\perp \mathcal{E}}{e c^2 B} = \frac{p_\perp}{e B}$$

one finds

$$m = \frac{1}{2} \frac{e^2 c^2}{\mathcal{E}} B R^2$$

$$= \frac{1}{2} \frac{c^2 p_\perp^2}{\mathcal{E} B}$$

Now let us show that the combination,

$$B R^2 = \frac{p_\perp^2}{e^2 B}$$

is an adiabatic invariant, i.e. it does not change upon slow variations of the parameters of the system (external magnetic field B in our case).

Consider first the case when the particle is moving along a circular orbit in a magnetic field slowly varying in time. The change of magnetic field flux through our contour induces the f.e.m. which changes the energy of a particle. The change in a period is

$$\Delta \mathcal{E} = e \, \pi R^2 \frac{dB}{dt}$$

$$\frac{\Delta \mathcal{E}}{T} \sim \frac{d\mathcal{E}}{dt} = \frac{1}{2} e \omega_B R^2 \frac{dB}{dt}$$

Consider now the change with time of the quantity

$$\frac{e^2}{\pi} \Phi = e^2 B R^2 = \frac{P_\perp^2}{B} = \frac{\mathcal{E}^2 - \mathcal{E}_0^2}{c^2 B}$$

magnetic flux through our contour

$\mathcal{E}_0 = mc^2$ = rest energy of a particle

$$\frac{d}{dt}\left(\frac{p^2}{B}\right) = \frac{1}{B}\,\frac{d}{dt}(p^2) - \frac{p^2}{B^2}\,\frac{dB}{dt} =$$

$$= \frac{1}{B\,c^2}\cdot 2\mathcal{E}\,\frac{d\mathcal{E}}{dt} - \frac{p^2}{B^2}\,\frac{dB}{dt} =$$

$$= \frac{1}{B c^2}\,\cancel{2}\mathcal{E}\,\frac{1}{\cancel{2}}\,e\,\omega_B\,R^2\,\frac{dB}{dt} - \frac{p^2}{B^2}\cdot\frac{dB}{dt} =$$

$$= \left\{ \frac{\cancel{e}\,\cancel{\mathcal{E}}}{\cancel{B}\,\cancel{c^2}}\;\frac{\cancel{e}\,\cancel{c^2}\,\cancel{B}}{\cancel{\mathcal{E}}}\;\frac{p^2}{\cancel{e^2}B^2} - \frac{p^2}{B^2} \right\} \frac{dB}{dt} =$$

$$= \left\{ \frac{p^2}{B^2} - \frac{p^2}{B^2} \right\} \frac{dB}{dt} = 0 . \qquad (*)$$

$\Rightarrow$ Thus, the flux of magnetic field throug particle orbit is indeed constant.

Note that we did not use the fact that the magnetic field itself is varying with time, only that the <u>flux</u> of magnetic field through the particle orbit changes. Thus, our calculation is applicable also to the case when the particle orbit, being circular, moves slowly through the space-dependent magnetic field.

Note also that our calculation is applicable in the case when the particle has the momentum parallel to the magnetic field. In this case in eq. (169*) one has to replace the momentum by the component perpendicular to the magnetic field. Thus we find

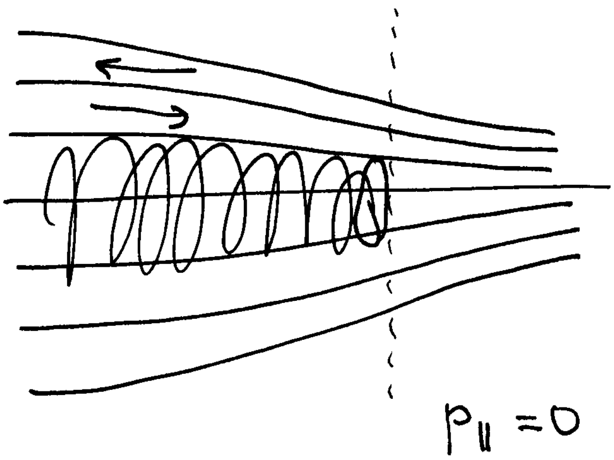
$$\frac{P_\perp^2}{B} = \text{const} \qquad \Rightarrow \quad P_\perp \propto \sqrt{B} \qquad (*)$$

in slowly-varying magnetic field.

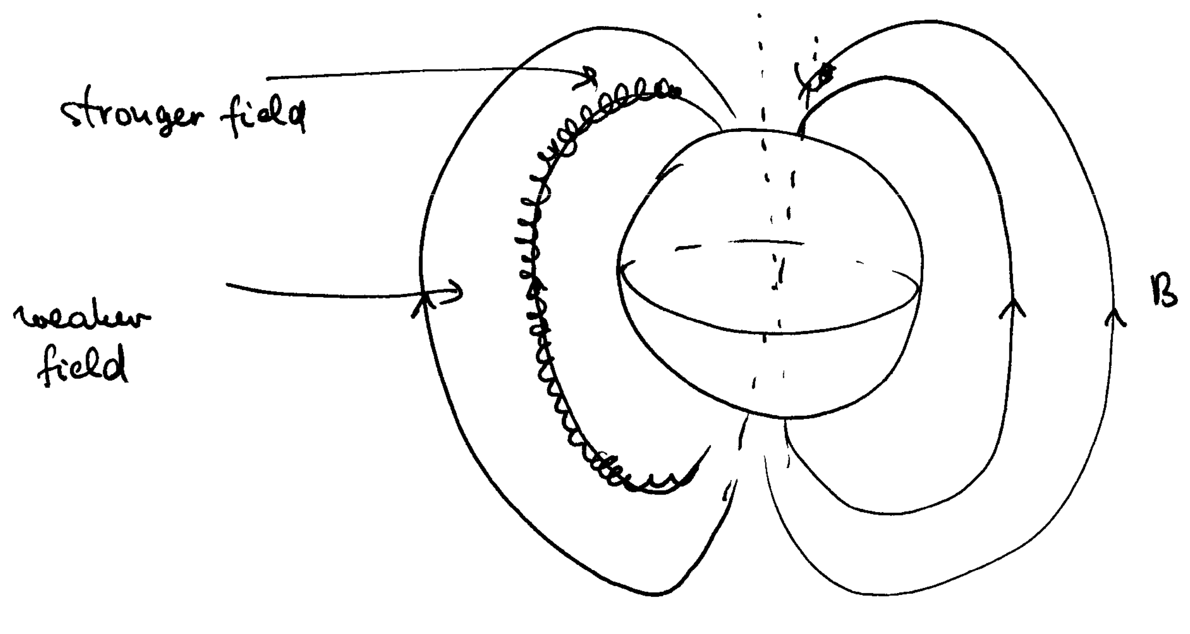
Physically, the two cases — when $\vec{B}$ depends on t and when a particle moves in a space-dependent, but time-independent magnetic field — are different. In the first case the total energy of the particle changes with time. In the second case when the magnetic field is static, the total energy of a particle <u>is constant</u>. Therefore, when $P_\perp$ changes, the longitudinal component of momentum also <u>must change</u>, so that $\vec{p}^{\,2} = \text{const}$. Thus,

$$p_{\parallel}^2 = \vec{p}^{\,2} - p_{\perp}^2 = \vec{p}^{\,2} - \text{const} \cdot |\vec{B}(\vec{x})|$$

This leads to <u>magnetic reflection</u> from regions of a strong magnetic field

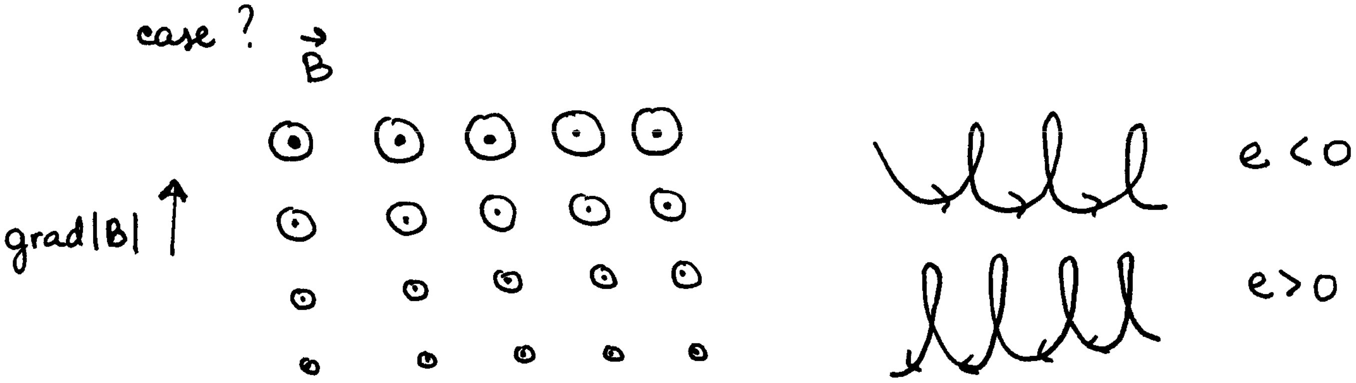


This trapping of charged particles happens, for instance, in the Earth magnetic field:



* Drift in non-uniform magnetic field.

Consider now the case when the magnetic field changes in the direction perpendicular to the magnetic field itself. How does the motion of a charged particle change in that case?



Qualitatively: one expect particles to drift (apart from their circular motion) in the direction perpendicular to both the magnetic field and its gradient. Particles with different charge drift in opposite directions, see the figure.

Let us calculate the drift velocity in the case when the magnetic field changes slowly, i.e., when the distance at which the change of magnetic field is of order one is much larger than the gyration radius of a particle. Mathematically,

$$\frac{a}{B}\left|\frac{dB}{dx^i}\right| \ll 1. \qquad (*)$$

Since there is no electric field, the motion is described by the equation

$$\frac{d\vec{v}}{dt} = \vec{v} \times \vec{\omega}_B(x); \qquad \left[\vec{\omega}_B(x) = \frac{e\vec{B}(x)}{m\gamma}\right] \qquad (**)$$

We suppose for simplicity that the direction of $\vec{B}$ does not change. Then $\vec{v}_\parallel$ drops out of the equation ($\vec{v}_\parallel = \text{const}$), and we can consider the orthogonal component only.

We solve the equations of motion approximately, making use of the relation $(*)$. To this end we expand the magnetic field around a given point (assume this is the origin):

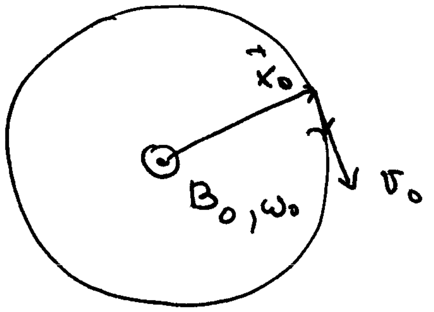
$$B(x) = B_0 + \vec{\nabla} B \cdot \vec{x} + \ldots$$
$$= B_0 \left(1 + \frac{\vec{\nabla} B \cdot \vec{x}}{B_0} + \ldots\right)$$

(here we denote $|\vec{B}(x)| = B(x)$). This has to be substituted into the equation (173 **):

$$\frac{d\vec{v}}{dt} = \vec{v} \times \vec{\omega}_0 \left(1 + \frac{1}{B_0} \vec{\nabla} B \cdot \vec{x} + \ldots\right) \qquad (*)$$

Now let us expand the motion into an unperturbed motion around the circular orbit, and correction:

$$\vec{x}(t) = \vec{x}_0(t) + \vec{x}_1(t)$$
$$\vec{v}(t) = \vec{v}_0(t) + \vec{v}_1(t) \qquad (**)$$



For the unperturbed motion we have the relations

$$\vec{v}_0 = -\vec{\omega}_0 \times \vec{x}_0$$

(see fig.) and also

$$\vec{\omega}_0 \times \vec{v}_0 = \omega_0^2 \, \vec{x}_0$$

Now substitute the relations (174**) into equation (174*). We have

$$\frac{d\vec{v}_0}{dt} + \frac{d\vec{v}_1}{dt} = \left(\vec{v}_0 + \vec{v}_1\right) \times \omega_0 \left(1 + \underbrace{\frac{1}{B_0}\vec{\nabla}B(\vec{x}_0 + \vec{x}_1) + \dots}_{\text{small correction}}\right)$$

Note: $\frac{\vec{\nabla}B \cdot x_0}{B_0} \sim \frac{\nabla B \cdot a}{B_0} \ll 1$.

Leading order:

$$\frac{d\vec{v}_0}{dt} = \vec{v}_0 \times \vec{\omega}_0 \qquad \rightarrow \text{ circular motion}$$

Next order:

$$\frac{d\vec{v}_1}{dt} = \vec{v}_1 \times \vec{\omega}_0 + \vec{v}_0 \times \vec{\omega}_0 \cdot \frac{1}{B}\vec{\nabla}B \cdot \vec{X}_0 +$$

$$= \left(\vec{v}_1 + \vec{v}_0 \, \frac{1}{B_0}\vec{\nabla}B \cdot \vec{X}_0\right) \times \vec{\omega}_0$$

Now use the relations that we have derived for the unperturbed motion:

$$\vec{v}_0 = -\vec{\omega}_0 \times \vec{X}_0$$

$$\frac{d\vec{v}_1}{dt} = \vec{v}_1 \times \vec{\omega}_0 - \frac{1}{B_0}(\vec{\nabla}B \cdot \vec{x}_0)\; \vec{\omega}_0 \times \vec{x}_0 \times \vec{\omega}_0 =$$

$$= \left[\vec{v}_1 - \vec{\omega}_0 \times \vec{x}_0 \cdot \frac{1}{B_0} \cdot (\vec{\nabla}B \cdot \vec{x}_0) \right] \times \vec{\omega}_0$$

This is a complicated equation, but we are not interested in the exact solution. Rather we are interested in the motion averaged over one or several periods. The l.h.s. averages to zero. Thus,

$$\langle \vec{v}_1 \rangle = \vec{\omega}_0 \times \left\langle \frac{1}{B_0} \vec{x}_0 \cdot (\vec{\nabla}B \cdot \vec{x}_0) \right\rangle$$

here $\langle f \rangle$ denotes average over a long period of time,

$$\langle f \rangle = \frac{1}{T} \int_0^T dt\; f(t) \; .$$

Since we know everything about the motion $\vec{x}_0(t)$, we can calculate the average,

$$\left\langle \frac{1}{B_0} \vec{x}_0 \cdot (\vec{\nabla}B \cdot \vec{x}_0) \right\rangle_i = \frac{1}{B_0} \nabla_j B \cdot \langle x_{0i}\, x_{0j} \rangle$$

$$X_{o1}(t) = a\cos(\omega_o t + \alpha)$$

$$X_{o2}(t) = a\sin(\omega_o t + \alpha)$$

Since

$$\langle \cos^2(\omega t + \alpha)\rangle = \frac{1}{2}$$

$$\langle \sin^2(\omega t + \alpha)\rangle = \frac{1}{2}$$

$$\langle \cos(\omega t + \alpha)\sin(\omega t + \alpha)\rangle = 0$$

we have

$$\langle X_{oi} X_{oj}\rangle = \frac{1}{2}\delta_{ij}$$

Thus,

$$\left\langle \frac{1}{B_o} X_{oi}\left(\vec{\nabla} B \cdot \vec{X}_o\right)\right\rangle = \frac{a^2}{2}\cdot\frac{1}{B_o}\cdot \nabla_i B$$

Making use of this relation we find

$$v_{drift} = \langle|\vec{v}_1|\rangle = \frac{\omega_o a^2}{2B_o}|\nabla B|$$

or, in vector form

$$\boxed{\vec{v}_{drift} = \frac{a^2}{2B_o}\cdot \vec{\omega}_o \times \vec{\nabla} B} \qquad (*)$$

We see from this equation that the direction of the drift is perpendicular to both the direction of the field, and the direction of the gradient.

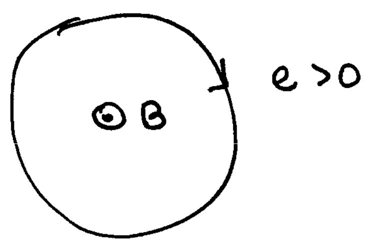
Eq. (177*) can be rewritten as

$$|\vec{v}_{drift}| = \frac{1}{2} \underbrace{a\,\omega_0}_{v_0} \cdot \underbrace{\frac{a\,|\nabla B|}{B_0}}_{\text{small parameter}}$$

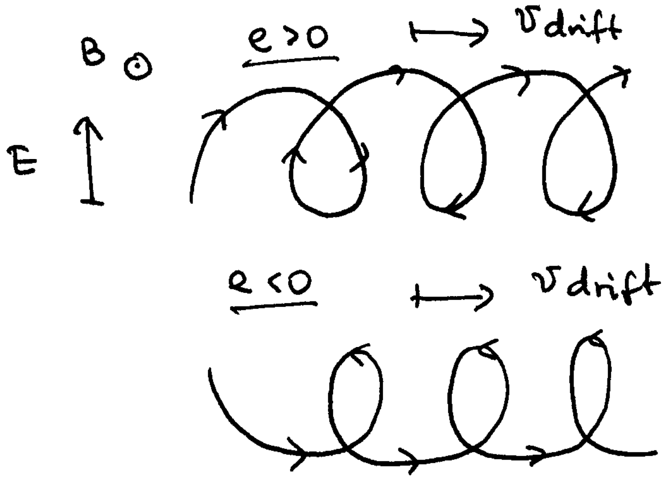
This shows that the drift velocity is much smaller than the velocity of the circular motion of the particle.

* Motion in a constant electric and magnetic field.

Consider first qualitatively the case of perpendicular electric and magnetic fields



← no electric field.



When moving in the direction of electric field, the particle accelerates (R becomes bigger). When moving against the electric field, the particle decelerates → radius of the trajectory becomes smaller.

⇒ The motion is similar to the drift in the inhomogeneous magnetic field, except particles of different signs drift in the same directions.

Consider the motion of the particle in more detail. For simplicity, assume that the particle is <u>non-relativistic</u>,

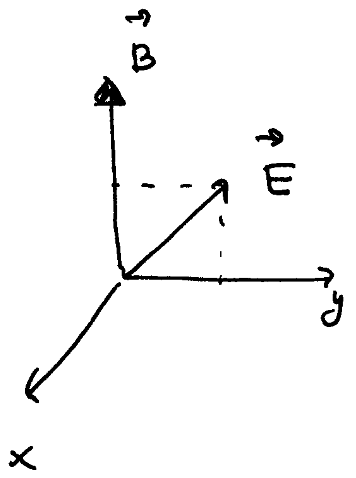
$$v \ll c,$$

$$\vec{p} \simeq m\vec{v}.$$

The equation of motion is then

$$m\dot{\vec{v}} = e\vec{E} + e[\vec{v} \times \vec{B}]$$

Let us choose the coordinates in such a way that $\vec{B} \parallel \vec{z}$, $\vec{B}$ and $\vec{E}$ lie in the plane (yz).



Then eqs. take the form

$$m\frac{d^2x}{dt^2} = e\frac{dy}{dt}\cdot B$$

$$m\frac{d^2y}{dt^2} = eE_y - e\frac{dx}{dt}\cdot B$$

$$m\frac{d^2z}{dt^2} = eE_z$$

The simplest of these equations is the third one. It describes a uniformly accelerated motion,

$$z(t) = \frac{1}{2} \frac{eE_z}{m} t^2 + v_{oz} \cdot t$$

Consider the equations for x and y. Multiply second by i and add:

$$m \frac{d^2}{dt^2} (x + iy) = ie E_y + eB \left(\frac{dy}{dt} - i \frac{dx}{dt} \right)$$

$$\frac{d}{dt} (\dot{x} + i\dot{y}) = i \frac{e}{m} E_y - i\omega (\dot{x} + i\dot{y})$$

$$\hookrightarrow \omega = \frac{eB}{m}$$

Thus, we have the equation

$$\frac{d}{dt} (\dot{x} + i\dot{y}) + i\omega (\dot{x} + i\dot{y}) = i \frac{e}{m} E_y \qquad (*)$$

$\Rightarrow$ inhomogeneous equation for $(\dot{x} + i\dot{y})$

solution to the homogeneous equation is

$$\dot{x} + i\dot{y} = A e^{-i\omega t}$$

where A is a complex constant. We choose it to be $A = a\,e^{i\alpha}$, then the phase can always be eliminated by a proper choice of the beginning of time, t_0. A particular solution of the inhomogeneous equation is

$$\dot{x} + i\dot{y} = \frac{eE_y}{m\omega} .$$

Combining these solution, one gets a general solution of eq. (181*):

$$\dot{x} + i\dot{y} = a\,e^{-i\omega t} + \frac{eE_y}{m\omega}$$

$$= a\,e^{-i\omega t} + \frac{E_y}{B}$$

From this expression we find

$$v_x = \dot{x} = a \cdot \cos(\omega t) + \frac{E_y}{B} \qquad (*)$$

$$v_y = \dot{y} = -a \cdot \sin(\omega t)$$

We see that the particle velocity contains an oscillating and a constant term. Averaging over time we get

$$\langle v_x \rangle = \frac{E_y}{B}$$

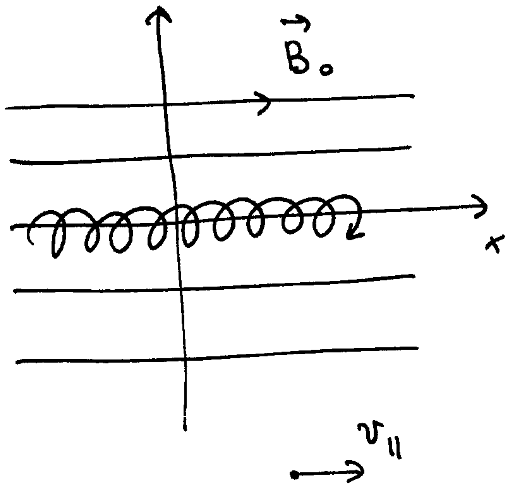
$$\langle v_y \rangle = 0$$

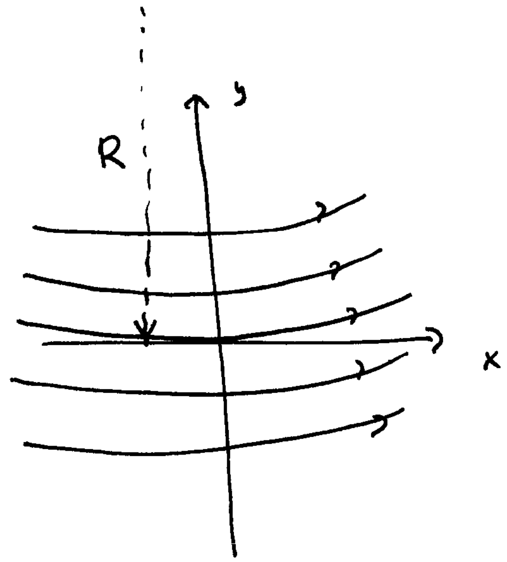
This averaged velocity is the <u>drift</u> velocity. In the vector form one may write it as

$$\vec{v}_{drift} = \frac{\vec{E} \times \vec{B}}{B^2}$$

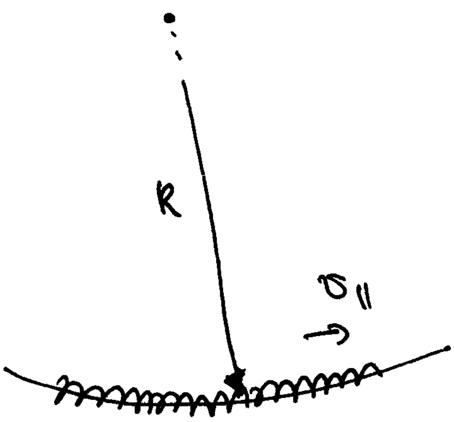
<u>Note</u> that this velocity does not depend on particle charge and mass. $\Rightarrow$ If we put plasma (a gas of positive and negative particles) in a constant electric and magnetic field, it will move as a whole in the direction of $\vec{E} \times \vec{B}$

* Drift of a particle due to curvature of magnetic field lines.





Consider a particle that spirals aroun magnetic field lines along orbits of radius a.
The situation when magnetic field lines are straight we have already discussed. The case of interest now is when the magnetic field lines are curved with radius $R \gg a$.



As far as the guiding center of the orbit is concerned, the motion is along the curved orbit of radius R with velocity $v_{\parallel}$.

Such a motion will produce a centrifugal acceleration of magnitude $v_\parallel^2/R$, which may be viewed as arising from an effective electric field

$$E_{eff} = \frac{\gamma m}{e} \cdot \frac{\vec{R}}{R^2} \cdot v_\parallel^2$$

The effect of electric field we have already considered; it leads to the drift which is expressed as $\vec{v} = \dfrac{\vec{E} \times \vec{B}}{B^2}$

or, in our case,

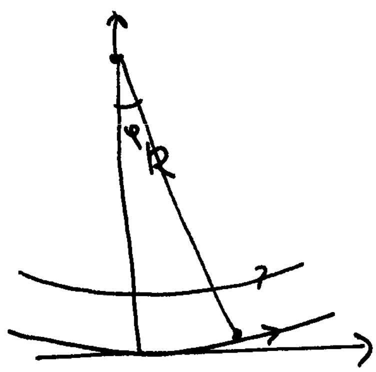
$$\vec{v}_c = \frac{\gamma m}{e} v_\parallel^2 \cdot \frac{\vec{R} \times \vec{B}_0}{R^2 \cdot B_0^2} = \frac{v_\parallel^2}{\omega_B R} \cdot \frac{\vec{R} \times \vec{B}_0}{R \cdot B_0} \qquad (*)$$

This equation can be, of course, rederived directly from the Lorentz equations (see problem).

The curvature drift, eq. (185*), and the gradient drift which we have considered before,

$$\vec{v}_G = \frac{a^2}{2B_0}\,\vec{\omega}_0 \times \vec{\nabla} B$$

can be combined <u>in a single expression</u>.



To see this, let us first rewrite the expression $\vec{\nabla} B$ in a different way.

Consider the magnetic field which is circular and changes in magnitude as a function of radius. For this field $B_z = 0$. Assume that there are no currents in the region. Then we have

$$\vec{\nabla} \times \vec{B} = 0$$

In components:

$$\partial_x B_y - \partial_y B_x = 0$$
$$\partial_y B_z - \partial_z B_y = 0$$
$$\partial_z B_x - \partial_x B_z = 0$$

Since $B_z = 0$ we have

$$\partial_x B_y - \partial_y B_x = 0 \qquad (*)$$

$$\partial_z B_y = 0$$

$$\partial_z B_x = 0$$

Now let us change variables according to fig. on page 186 :

$$x = \rho \sin\varphi$$

$$y = R - \rho\cos\varphi$$

$$dx = d\rho \cdot \sin\varphi + \rho\cos\varphi \, d\varphi$$

$$dy = -d\rho \cos\varphi + \rho\sin\varphi \, d\varphi$$

$$d\rho = \sin\varphi \, dx - \cos\varphi \, dy \equiv \frac{\partial\rho}{\partial x}dx + \frac{\partial\rho}{\partial y}dy$$

$$d\varphi = \frac{1}{\rho}\left(\cos\varphi \cdot dx + \sin\varphi \, dy\right) \equiv \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy$$

Our magnetic field has the forme

$$B_x = B(\rho) \cdot \cos\varphi$$

$$B_y = B(\rho) \cdot \sin\varphi$$

$$B_z = 0 .$$

Now we can calculate the first of eqs. (187*) :

$$0 = \partial_x B_y - \partial_y B_x =$$

$$= B' \partial_x \rho \cdot \sin\varphi + B \cos\varphi \, \partial_x \varphi$$

$$- B' \partial_y \rho \cdot \cos\varphi + B \sin\varphi \, \partial_y \varphi =$$

$$= B' \sin^2\varphi + \frac{B}{\rho} \cos^2\varphi +$$

$$+ B' \cos^2\varphi + \frac{B}{\rho} \sin^2\varphi =$$

$$= B' + \frac{B}{\rho} = 0 .$$

We see that the gradient of B is related to its value. In the vector form this relation can be written as (setting $\rho = R$):

$$\frac{\vec{\nabla}_\perp B}{B} = - \frac{\vec{R}}{R^2}$$

Making use of this relation and the relation $a = \frac{v_\perp}{\omega_B}$ we can rewrite $\vec{v}_G$ as follows:

$$\vec{v}_G = \frac{1}{2} \frac{a^2}{B_0} \vec{\omega}_B \times \nabla \vec{B} =$$

$$= \frac{1}{2} \frac{v_\perp^2}{\omega_B^2} \frac{1}{B_0} \underbrace{\omega_B \cdot \frac{\vec{B}}{B_0}}_{\vec{\omega}_B} \times \left(- \frac{\vec{R}}{R^2} \cdot B_0 \right)$$

$$= \frac{1}{2} v_\perp^2 \cdot \frac{1}{\omega_B \cdot R} \cdot \frac{\vec{R} \times \vec{B}}{R B_0}$$

Now this equation can be combined with the one for the drift in curved magnetic field lines, (185*) :

$$\boxed{\vec{v}_D = \vec{v}_G + \vec{v}_C = \frac{1}{\omega_B R} \left(v_\parallel^2 + \frac{1}{2} v_\perp^2 \right) \frac{\vec{R} \times \vec{B}}{R \cdot B_0}}$$

Numerically, magnitude of the drift velocity is

$$v_D = 172 \cdot \left(\frac{T}{1 k^\circ} \right) \cdot \left(\frac{1 m}{R} \right) \cdot \left(\frac{1 G}{B} \right) \cdot \frac{cm}{s}$$

This parameter is important for thermonuclear reactions where the problem is to confine hot plasma for sufficiently long time. Due to the drift phenomenon charged particles move out of the confinement area and hit the walls. For typical parameters

$$R \sim 1\,\text{m}$$
$$B \sim 10^3\ \text{G}$$
$$T \sim 10^4\ \text{K} \quad (\sim 1\,\text{eV})$$

the drift velocity is

$$v_D \sim 2\cdot 10^3\ \text{cm/s}.$$

* Field invariants

We have considered so far several cases of motion of a charged particle in electromagnetic field. However, our original equation (155 *) is Lorentz-covariant, i.e. it can be considered in any frame. The question thus arises of whether the cases we have considered are different or they can be transformed to one another by Lorentz transformations?

Rather than trying to answer this question by deriving the transformation rules of $\vec{E}$ and $\vec{B}$ (see Jackson), consider quantities that are invariant.

We have seen that electric and magnetic fields are parts of a 4-tensor $F^{\mu\nu}$:

$$F^{i0} = E_i$$

$$F^{ij} = -c\,\epsilon_{ijk} B_k$$

(recall that $F^{\mu\nu}$ is antisymmetric, $F^{\mu\nu} = -F^{\nu\mu}$).

In terms of $F^{\mu\nu}$ invariants are easy to construct. We just have to build a scalar by contracting $F^{\mu\nu}$, $\eta_{\mu\nu}$ and $\epsilon_{\mu\nu\lambda\rho}$ (absolutely antisymmetric tensor).

1) $F^{\mu\nu}\eta_{\mu\nu} = 0 \Rightarrow$ does not work.

2) $F^{\mu\nu}F^{\lambda\rho}\eta_{\mu\lambda}\eta_{\nu\rho} = F^{\mu\nu}F_{\mu\nu}$ – ok.

3) $F^{\mu\nu}F^{\lambda\rho}\epsilon_{\mu\nu\lambda\rho}$ – ok.

Higher powers of $F^{\mu\nu}$ give nothing new. Consider first the second case:

$$F^{\mu\nu}F_{\mu\nu} = \underbrace{F^{0i}}_{E^i} F_{0i} + \underbrace{F^{i0}}_{E^i} \underbrace{F_{i0}}_{-E_i} + F^{ij}F_{ij} =$$

$$= -2\vec{E}^2 + c^2 \underbrace{\epsilon_{ijk} B_k \epsilon_{ijq}}_{2\delta_{kq}} B_q =$$

$$= 2\left\{c^2\vec{B}^2 - \vec{E}^2\right\}$$ – this is one of the invariants.

Consider now 3):

$$F^{\mu\nu} F^{\lambda\rho} \cdot \epsilon_{\mu\nu\lambda\rho} = 4 \cdot F^{0i} F^{jk} \quad \epsilon_{0ijk}$$

$$= 4 \cdot E_i \cdot \left(-c\, \epsilon_{jkp} B_p \right) \epsilon_{ijk} =$$

$$= -4c \cdot 2\, E_i B_i = -8c \underbrace{\vec{E} \cdot \vec{B}}$$

this is another invariant.

Thus, we conclude that there are 2 Lorentz-invariants which may be composed of the fields $\vec{E}$ and $\vec{B}$:

$$I_1 = c^2 \vec{B}^2 - \vec{E}^2$$

$$I_2 = \vec{E} \cdot \vec{B}$$

Invariance of these two quantities implies a number of useful relations. In particular, if $\vec{E} \perp \vec{B}$ in one frame, the same is true for all frames. If $|cB| \gtreqless |E|$ in one frame, it is true in all frames.

ELECTROMAGNETIC WAVES

So far we have considered electric and magnetic fields created by charges, either static or moving. But it turns out that electromagnetic fields may exist without any sources (charges) that produce them. So let us go back to Maxwell equations with the r.h.s. (the source part) set to zero. We will see that there still exist solutions to these equations which are non-zero, the electromagnetic waves.

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho \longrightarrow \boxed{\vec{\nabla}\cdot\vec{E} = 0}$$

$$\vec{\nabla}\times\vec{B} - \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J} \longrightarrow \boxed{\vec{\nabla}\times\vec{B} - \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} = 0}$$

$$\boxed{\begin{aligned}\vec{\nabla}\times\vec{E} + \frac{\partial B}{\partial t} &= 0 \\ \vec{\nabla}\cdot\vec{B} &= 0\end{aligned}}$$

There is only one constant entering these equations, $\epsilon_0\mu_0$.

making use of the relations

$$\epsilon_0 = 8{,}8541878176 \cdot 10^{-12} \cdot \frac{C^2}{N m^2}$$

$$\mu_0 = 4\pi \cdot 10^{-7} \frac{N s^2}{C^2}$$

we find that $(\epsilon_0 \mu_0)^{-1}$ has dimension of $(\text{velocity})^2$,

$$(\epsilon_0 \mu_0)^{-1} = 8{.}9875517873 9 \cdot 10^{16} \frac{m^2}{s^2} =$$

$$= \left(299792458 \frac{m}{s}\right)^2 = c^2$$

In terms of c, the Maxwell equations read

$$\vec{\nabla} \vec{E} = 0$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = 0 \qquad (*)$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

Let us solve these equations. We will be looking for solutions that are time-dependent (time-independent ones have been considered in the first part, electrostatics).

Take curl of the third equation:

$$\underbrace{\varepsilon_{ijk}\,\partial_j\,\varepsilon_{kmn}}\,\partial_m E_n + \varepsilon_{ijk}\,\partial_j \frac{\partial B_k}{\partial t} = 0$$

$$\varepsilon_{ijk}\,\varepsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

$$(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\,\partial_j\partial_m E_n + \frac{\partial}{\partial t}\cdot\underbrace{\varepsilon_{ijk}\,\partial_j B_k}_{\frac{1}{c^2}\frac{\partial E_i}{\partial t}} = 0$$

(second eq.)

$\delta_{jn}\,\partial_j E_n = \vec{\nabla}\cdot\vec{E} = 0.$ (first eq).

$$-\partial_j^2 E_i + \frac{\partial}{\partial t}\cdot\left(+\frac{1}{c^2}\frac{\partial E_i}{\partial t}\right) = 0$$

$$\boxed{\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \partial_j^2\right]E_i = 0.}$$

Alternatively, taking curl of the second equation we have

$$\varepsilon_{ijk}\,\partial_j\,\varepsilon_{kmn}\,\partial_m B_n - \frac{1}{c^2}\frac{\partial}{\partial t}\,\varepsilon_{ijk}\,\partial_j E_k = 0$$

$$\boxed{\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \partial_j^2\right]B_i = 0}$$

Thus, we see that all components of $\vec{E}$ and $\vec{B}$ satisfy the same equation which is called the <u>d'Alambert equation</u>.

* Eq. for electromagnetic potentials ϕ and $\vec{A}$.

The fields $\vec{E}$ and $\vec{B}$ are expressed in terms of ϕ and $\vec{A}$ as follows:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

These equations do not fix the potentials φ and $\vec{A}$ uniquely: the potentials

$$\phi' = \phi + \frac{\partial \lambda(t,x)}{\partial t}$$

$$\vec{A}' = \vec{A} - \vec{\nabla}\lambda(t,x)$$

produce the same $\vec{E}$ and $\vec{B}$ for any function $\lambda(t,\bar{x})$. One may use this freedom to set

$$\phi = 0$$

This does not fix $\lambda(t,x)$ completely; one may change it by a function which is only a function of $\vec{x}$, not t.

Now let us make use of the equation (195*),

$$\vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = 0$$

we have

$$\underbrace{\varepsilon_{ijk}\,\partial_j\,\varepsilon_{kmn}}\partial_m A_n + \frac{1}{c^2}\frac{\partial^2 A_i}{\partial t^2} = 0$$

$$\left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right)\partial_j\partial_m A_n =$$

$$= \partial_i\left(\partial_j A_j\right) - \partial_j^2 A_i$$

so we get the equation

$$\partial_i\left(\cancel{\partial_j A_j}\right) - \partial_j^2 A_i + \frac{1}{c^2}\frac{\partial^2 A_i}{\partial t^2} = 0$$

because of the gauge condition $\vec{\nabla}\cdot\vec{A} = 0$.

Note that at $\phi = 0$ equation $\bar{\nabla}\bar{E} = 0$ implies that

$$\frac{\partial}{\partial t}\bar{\nabla}\bar{A} = 0$$

and thus $\vec{\nabla}\vec{A}$ is time-independent. This is why it can be set to zero by a time-independent residual gauge transformation.

Finally, we have

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \partial_j^2\right]A_i = 0,$$

the same equation as for E_i and B_i.

Let us discuss now general solution to the d'Alambert equation, which has the form

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} - \partial_i^2 f = 0$$

Consider first the case when the solution depends only on one coordinate, say x. Such solution are called <u>plane waves</u>. The equation then becomes

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} = 0 .$$

Let us make the following change of variables:

$$\xi = ct - x$$

$$\eta = ct + x$$

Then
$$d\xi = c\,dt - dx$$
$$d\eta = c\,dt + dx$$

$$\partial_x = \frac{\partial \eta}{\partial x}\partial_\eta + \frac{\partial \xi}{\partial x}\partial_\xi = \partial_\eta - \partial_\xi$$

$$\frac{1}{c}\partial_t = \frac{1}{c}\frac{\partial \eta}{\partial t}\partial_\eta + \frac{1}{c}\frac{\partial \xi}{\partial t}\partial_\xi = \partial_\eta + \partial_\xi$$

The equation becomes

$$0 = \left(\frac{1}{c^2}\partial_t^2 - \partial_x^2\right) f = \left(\frac{1}{c}\partial_t - \partial_x\right)\left(\frac{1}{c}\partial_t + \partial_x\right) f$$

$$= 4\, \partial_\eta \partial_\xi \cdot f = 0$$

This equation is very easy to solve, obviously a general solution has the form

$$f = f_1(\xi) + f_2(\eta) =$$

$$= f_1(ct - x) + f_2(ct + x) \qquad (*)$$

where f_1, f_2 are two <u>arbitrary</u> functions.

The function $f_1(ct - x)$ describes a plane wave moving in the positive direction of the x-axis, while $f_2(ct + x)$ describes a wave moving in the negative direction.

Consider a wave that moves in the positive direction of x. In this wave all components of E_i and B_i (and of A_i) depend on $\xi = ct - x$. We still have to construct a solution to the original Maxwell equations of p. (195 *).

Consider first of eqs. (195*):

$$0 = \vec{\nabla}\vec{E} = \partial_x E_x(ct-x) + \cancel{\partial_y E_y(ct-x)} + \cancel{\partial_z E_z(ct-x)}$$

$$= -\frac{1}{c}\partial_t E_x$$

$$\Rightarrow \quad E_x = \text{const.}$$

We are not interested in this part of the solution because it represent the static fields which we have already studied. Thus, we set this constant to zero. The same argument leads to $B_x = 0$ from the last eq. (195*).

$\Rightarrow$ We see thus that in the propagating wave the directions of the fields $\vec{E}$ and $\vec{B}$ are perpendicular to the direction of propagation. For this reason EM waves are called <u>transverse</u>.

Denoting the propagation direction by $\vec{n}$, we therefore have

$$\vec{n}\cdot\vec{E} = 0 \quad ; \quad \vec{n}\cdot\vec{B} = 0$$

Consider now the other two equations. Since $\vec{n} = (1,0,0)$ is a unit vector in the x-direction, we can write

$$x = \vec{n}\cdot\vec{x} \qquad \left[\vec{x} = (x,y,z)\right]$$

Thus our fields depend on $\xi = ct - \vec{n}\vec{x}$,

$$\vec{E} = \vec{E}\,(ct - \vec{n}\vec{x})$$
$$\vec{B} = \vec{B}\,(ct - \vec{n}\vec{x})$$

Substituting these fields into eqs. (195*) we get

$$\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\partial_t\vec{E} = -\vec{n}\times\partial_\xi\vec{B} - \frac{1}{c^2}c\,\partial_\xi\vec{E} =$$

$$= -\partial_\xi\left\{\vec{n}\times\vec{B} + \frac{1}{c}\vec{E}\right\} = 0$$

$$\Rightarrow \qquad \vec{n}\times\vec{B} + \frac{1}{c}\vec{E} = \text{const}.$$

As before, we are interested in solutions which do depend on ξ, so we set the constant to 0. Thus,

$$\boxed{\vec{n}\times\vec{B} + \frac{1}{c}\vec{E} = 0} \qquad (*)$$

Similarly, we obtain from the third eq. (195*):

$$\boxed{\vec{n} \times \vec{E} - c\vec{B} = 0} \qquad (*)$$

This is, in fact, not a new equation, since we can express $\vec{E}$ from eq. (202*),

$$E = -c\,\vec{n} \times \vec{B}$$

and substitute into (*):

$$\underbrace{-\vec{n} \times (c\,\vec{n} \times \vec{B})} - c\vec{B} = c\vec{B} - c\vec{B} = 0 .$$

$$\epsilon_{ijk} n_j \, \epsilon_{kmn} n_m B_n = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) n_j n_m B_n =$$

$$= n_i \cancel{(\vec{n}\vec{B})} - \vec{n}^2 \cdot B_i =$$

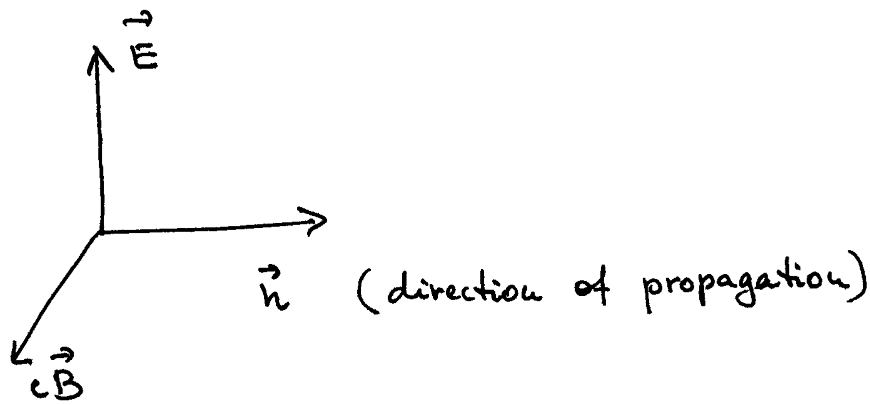
$$= -B_i$$

So, eq. (*) is not an independent one, provided we have imposed $\vec{n}\vec{B} = 0$. Note that multiplying eq. (202*) by $\vec{n}$ gives

$$\vec{n} \cdot (\vec{n} \times \vec{B}) + \frac{1}{c}\vec{n}\vec{E} = \frac{1}{c}\vec{n} \cdot \vec{E} = 0$$

which is consistent with our choice $\vec{n} \cdot \vec{E} = 0$.

From eq. (202*) we see that in the plane electromagnetic wave the vectors $\vec{B}$ and $\vec{E}$ are mutually orthogonal and both are also orthogonal to the direction of propagation.



We also see that the vectors $c\vec{B}$ and $\vec{E}$ have equal length.

Recall that two quantities built of electric and magnetic field are Lorentz-invariant,

$$c^2\vec{B}^2 - \vec{E}^2$$

$$\vec{B}\cdot\vec{E}$$

Both these quantities are equal to zero for the electromagnetic wave; this is true in any frame

$\Rightarrow$ <u>in any frame</u> the magnetic and electric fields of an electromagnetic wave are equal, $|c\vec{B}| = |\vec{E}|$, and perpendicular, $\vec{E}\cdot\vec{B} = 0$.

Monochromatic plane wave

An important particular case is periodic dependence on time with a single frequency,

$$\vec{E}, \vec{B} \propto e^{-i\omega t}.$$

In this case the d'Alampert equation takes the form

$$\left[\frac{\omega^2}{c^2} + \partial_i^2 \right] E_j = 0$$

$$\left[\frac{\omega^2}{c^2} + \partial_i^2 \right] B_j = 0$$

For plane waves, the solutions to these equations follow from eqs. (200*):

$$\vec{E} = \mathrm{Re}\left[\vec{\mathcal{E}}_0 \cdot e^{-i\omega t + i\vec{k}\vec{x}} \right]$$
$$\vec{B} = \mathrm{Re}\left[\vec{B}_0 \cdot e^{-i\omega t + i\vec{k}\vec{x}} \right] \qquad (*)$$

where $\vec{k} = \vec{n}\, \omega/c$:

$$i\omega t - i\vec{k}\vec{x} = i\frac{\omega}{c}\left(ct - \frac{c\vec{k}}{\omega}\vec{x}\right) = i\frac{\omega}{c}\left(ct - \vec{n}\vec{x}\right)$$

$\vec{n}$ = unit vector in the direction of the propagation.

The vector $k = \frac{\omega}{c}\vec{n}$ is called the <u>wave vector</u>. In eqs. (205*) $\vec{E}_0$ and $\vec{B}_0$ are two arbitrary complex vectors.

The plane monochromatic wave is periodic both in space and time. Consider electric (magnetic) field at a given point in space, $\vec{x} = \vec{x}_0 =$ fixed. After a time

$$T = \frac{2\pi}{\omega}$$

the value of the field returns to its original value (the argument of the exponent changes by $2\pi i$). Thus, T is called a period of the wave, while $f = 1/T = \frac{\omega}{2\pi}$ is its frequency.

Now consider space dependence of the field value at a given moment of time. As a function of x, the distance in the direction $\vec{n}$, the field value is periodic with period

$$\lambda = 2\pi \cdot \frac{c}{\omega}$$

This quantity is called <u>wavelength</u>.

For the monochromatic wave there is a relation between the wave vector $\vec{k}$, $\vec{E}$ and $\vec{B}$. To find this relation let us plug the expressions (205*) into the Maxwell equations. Since the equations are linear, we may drop the sign of the real part.

$$\vec{\mathcal{E}}_0 \cdot i\vec{k}\, e^{-i\omega t + i\vec{k}\vec{x}} = 0 \quad \Rightarrow \quad \boxed{\vec{k}\cdot\vec{\mathcal{E}}_0 = 0}$$

$$i\vec{k}\cdot\vec{B}_0\, e^{-i\omega t + i\vec{k}\vec{x}} = 0 \quad \Rightarrow \quad \boxed{\vec{k}\cdot\vec{B}_0 = 0}$$

$$i\vec{k}\times\vec{B}_0\, e^{-i\omega t + i\vec{k}\vec{x}} + i\frac{\omega}{c^2}\vec{\mathcal{E}}_0\, e^{-i\omega t + ikx} = 0$$

$$\boxed{\vec{k}\times\vec{B}_0 + \frac{\omega}{c^2}\vec{\mathcal{E}}_0 = 0}$$

$$i\vec{k}\times\vec{\mathcal{E}}_0\, e^{-i\omega t + ikx} - i\omega\vec{B}_0\, e^{-i\omega t + ikx} = 0$$

$$\boxed{\vec{k}\times\vec{\mathcal{E}}_0 - \omega\vec{B}_0 = 0}$$

These relations are particular cases of (202*) and (203*).

Polarization

The waves we have considered so far always have the electric field parallel to a given vector, $\vec{\epsilon}_1$. Such fields are called <u>linearly polarized</u> in the direction of $\vec{\epsilon}_1$. Obviously, a wave linearly polarized in the direction $\vec{\epsilon}_2$ which is orthogonal to $\vec{\epsilon}_1$ and the direction of propagation $\vec{k}$ is linearly independent. The two waves can be combined to give the most general monochromatic wave propagating in the direction $\vec{k}$,

$$\vec{E}(x,t) = \left(\vec{\epsilon}_1 \, \mathcal{E}_1 + \vec{\epsilon}_2 \, \mathcal{E}_2 \right) e^{-i\omega t + i\vec{k}\vec{x}} \qquad (*)$$

real mutually orthogonal unit vectors

complex amplitudes; their relative phase represents the phase difference between two waves

$$\vec{B}(x,t) = \vec{B}_0 \, e^{-i\omega t + i\vec{k}\vec{x}}$$

where

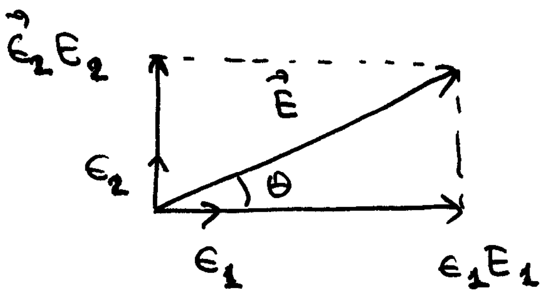
$$\vec{B}_0 = \frac{1}{\omega} \vec{k} \times \left(\vec{\epsilon}_1 \mathcal{E}_1 + \vec{\epsilon}_2 \mathcal{E}_2 \right)$$

Now recall that the physical (real) electric field is obtained by taking the real part of eq (208*). Therefore, if $\mathcal{E}_1$ and $\mathcal{E}_2$ have <u>the same phase</u>, then the resulting field is also <u>linearly polarized</u>, but in a different direction.

$$\mathcal{E}_1 = E_1 e^{i\varphi} \qquad\qquad E_1, E_2 = \text{real}$$

$$\mathcal{E}_2 = E_2 e^{i\varphi}$$

$$\vec{E}(x,t) = \underbrace{\left(\vec{\epsilon}_1 \cdot E_1 + \vec{\epsilon}_2 E_2\right)}_{\substack{\text{this sets amplitude and direction} \\ \text{of the resulting field}}} \cdot \cos\left\{\varphi - \omega t + \vec{k}\vec{x}\right\}$$



$$\operatorname{tg}\theta = \frac{E_2}{E_1}$$

$$|E| = \sqrt{E_1^2 + E_2^2}$$

On the contrary, if the complex amplitudes $\mathcal{E}_1$ and $\mathcal{E}_2$ have <u>different</u> phases, then the direction of the resulting electric field is constantly changing in time. Such waves are called <u>elliptically polarized</u>.

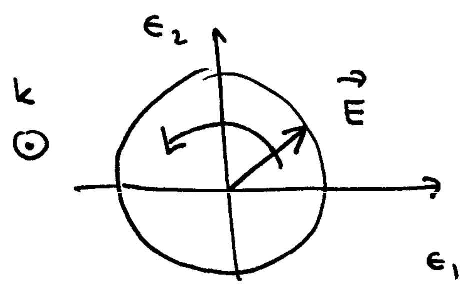
To understand why, consider first a particular case when $|\mathcal{E}_1| = |\mathcal{E}_2|$ and the phase differs by $\pi/2$ ($90°$). Then the actual (real) electric field is

$$\vec{E} = \mathrm{Re}\left[E_0 \cdot (\vec{\epsilon}_1 \pm i\vec{\epsilon}_2)\, e^{-i\omega t + i\bar{k}\bar{x}} \quad e^{i\varphi}\right] \qquad (*)$$

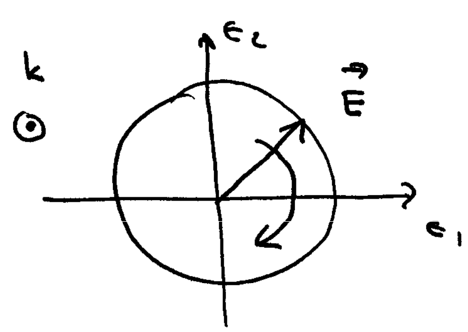
$$= E_0 \left\{ \vec{\epsilon}_1 \cos(\varphi - \omega t + \bar{k}\bar{x}) \mp \vec{\epsilon}_2 \sin(\varphi - \omega t + \bar{k}\bar{x}) \right\}$$

In a given point in space $\vec{x}_0$, this expression describes a vector of magnitude E_0 rotating with the frequency ω

$\vec{\epsilon}_1 + i\vec{\epsilon}_2$ $\qquad\qquad$ $\vec{\epsilon}_1 - i\vec{\epsilon}_2$



a)



b)

The waves described by eq. (210*) are called <u>circularly polarized</u>. The upper sign $\vec{\epsilon}_1 + i\vec{\epsilon}_2$ corresponds to the case <u>left circular polarization</u> (counterclockwise rotation of incoming wave). The opposite sign $\vec{\epsilon}_1 - i\vec{\epsilon}_2$ corresponds to right circular polarization.

left polarization = positive helicity

right polarization = negative helicity.

Instead of decomposing an arbitrary wave into superposition of linearly polarized waves, one may use complex polarization vectors that correspond to circular polarization. One defines

$$\vec{\epsilon}_{\pm} = \frac{1}{\sqrt{2}} \left(\vec{\epsilon}_1 \pm i\vec{\epsilon}_2\right)$$

such that

$$\vec{\epsilon}^{\,*}_{\pm} \; \vec{\epsilon}_{\mp} = 0$$

$$\vec{\epsilon}^{\,*}_{\pm} \cdot \mathbf{k} = 0$$

$$\epsilon^{*}_{\pm} \; \epsilon_{\pm} = 1$$

Then a general field of a plane wave is decomposed as

$$\vec{E}(\bar{x},t) = \left(E_+ \vec{\epsilon}_+ + E_- \vec{\epsilon}_- \right) e^{-i\omega t + i k \bar{x}}$$

let us see what behavior of the electric field $\vec{E}$ this equation corresponds to. Denote

$$\theta = -\omega t + k\bar{x}$$

$$E_+ = \text{real}$$

$$E_- \Rightarrow e^{i\alpha}\, r \cdot E_+$$

so that

$$\frac{E_-}{E_+} = r e^{i\alpha} = r\left(\cos\alpha + i \sin\alpha\right)$$

Then

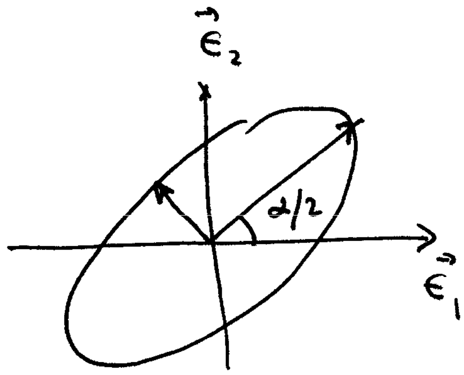
$$\vec{E} = \frac{E_+}{\sqrt{2}} \mathrm{Re} \left\{ \left((\vec{\epsilon}_1 + i \vec{\epsilon}_2) + r(\cos\alpha + i \sin\alpha)(\vec{\epsilon}_1 - i \vec{\epsilon}_2) \right) \times \right.$$
$$\left. \times \left(\cos\theta + i \sin\theta\right) \right\}$$

$$= \frac{E_+}{\sqrt{2}} \mathrm{Re} \left\{ \left(\vec{\epsilon}_1 + r\cos\alpha\, \vec{\epsilon}_1 + r \sin\alpha\, \vec{\epsilon}_2 + \right. \right.$$
$$\left. + i\left[\vec{\epsilon}_2 + r \sin\alpha\, \vec{\epsilon}_1 - r\cos\alpha\, \vec{\epsilon}_2 \right] \right) \times$$
$$\left. \times \left(\cos\theta + i \sin\theta\right) \right\}$$

$$= \frac{E_+}{\sqrt{2}} \Big\{ (\vec{\epsilon}_1 + r\cos\alpha\,\vec{\epsilon}_1 + r\sin\alpha\,\vec{\epsilon}_2)\cos\theta +$$

$$+ \left(-\vec{\epsilon}_2 - r\sin\alpha\,\vec{\epsilon}_1 + r\cos\alpha\,\vec{\epsilon}_2\right)\sin\theta \Big\}$$

This equation describes a rotating vector whose extremety moves along the ellips with the semi-axes $\frac{E_+}{\sqrt{2}}|(1+r)|$ and $\frac{E_+}{\sqrt{2}}|(1-r)|$ and with semi-major axis turned with respect to $\vec{\epsilon}_1$ by an angle $\alpha/2$:



The ratio of the axes is

$$\left| \frac{1-r}{1+r} \right|$$

so that for $r=1$ we get back to the linear polarization, and for $r=0$ → to the circular polarization.

Stokes parameters (Stokes' 1852)

In practice, one often has to determine the polarization state (i.e., the coefficients E_1, E_2 or E_+, E_-) for a given wave $\vec{E}$. For this purpose one introduces the (experimentally measurable) Stokes parameters defined as follows:

For linear basis:

$$S_0 = |\vec{\epsilon}_1 \cdot \vec{E}|^2 + |\vec{\epsilon}_2 \cdot \vec{E}|^2 = a_1^2 + a_2^2$$

$$S_1 = |\vec{\epsilon}_1 \cdot \vec{E}|^2 - |\vec{\epsilon}_2 \cdot \vec{E}|^2 = a_1^2 - a_2^2$$

$$S_2 = 2\,\mathrm{Re}\left[(\vec{\epsilon}_1 \cdot \vec{E})^* (\vec{\epsilon}_2 \cdot \vec{E})\right] = 2 a_1 a_2 \cos(\delta_2 - \delta_1)$$

$$S_3 = 2\,\mathrm{Im}\left[(\vec{\epsilon}_1 \cdot \vec{E})^* (\vec{\epsilon}_2 \cdot \vec{E})\right] = 2 a_1 a_2 \sin(\delta_2 - \delta_1)$$

where
$$E_1 = a_1 e^{i\delta_1}$$
$$E_2 = a_2 e^{i\delta_2}$$

The same parameters may be defined in terms of circularly polarized vectors. Denoting

$$E_+ = a_+ e^{i\delta_+}$$

$$E_- = a_- e^{i\delta_-}$$

$$S_0 = |\vec{\epsilon}_+^{\,*} \vec{E}|^2 + |\vec{\epsilon}_-^{\,*} \vec{E}|^2 = a_+^2 + a_-^2$$

$$S_1 = 2\,\mathrm{Re}\left[(\vec{\epsilon}_+^{\,*} \cdot \vec{E})^* \cdot (\vec{\epsilon}_-^{\,*} \vec{E})\right] = 2 a_+ a_- \cos(\delta_- - \delta_+)$$

$$S_2 = 2\,\mathrm{Im}\left[(\vec{\epsilon}_+^{\,*} \cdot \vec{E})^* \cdot (\vec{\epsilon}_-^{\,*} \vec{E})\right] = 2 a_+ a_- \sin(\delta_- - \delta_+)$$

$$S_3 = |\vec{\epsilon}_+^{\,*} \vec{E}|^2 - |\vec{\epsilon}_-^{\,*} \cdot \vec{E}|^2 = a_+^2 - a_-^2$$

One uses sometimes the notations I, Q, U, V.

The four parameters are not independent as they depend on only 3 quantities a_1, a_2 and $\delta_2 - \delta_1$. They satisfy the relation

$$S_0^2 = S_1^2 + S_2^2 + S_3^2$$

Energy and momentum of electromagnetic wave. Poynting's theorem (Poynting' 1884)

Let us establish the energy conservation for a system involving electromagnetic field (not necessarily static). If the system involves charges that move, the work performed on these charges in a unit time is $\sum_i q_i \vec{v}_i \vec{E}$ (only the electric field performs the work). For a continuous charge distribution this becomes

$$\begin{pmatrix}\text{rate of} \\ \text{work}\end{pmatrix} = \int_V \vec{J}\cdot\vec{E}\, d^3x .$$

This must be equal to the decrease of energy of the electromagnetic field. Indeed, from the equation

$$\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

we have

$$\int_V d^3x \; \vec{J}\cdot\vec{E} = \int_V d^3x \; \vec{E} \left\{ \frac{1}{\mu_0} \vec{\nabla}\times\vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right\}$$

Now consider the vector identity

$$\vec{E}\cdot\vec{\nabla}\times\vec{B} = \varepsilon_{ijk} E_i \partial_j B_k =$$

$$= \varepsilon_{ijk} \left(\partial_j (E_i B_k) - B_k \partial_j E_i \right) =$$

$$= -\vec{\nabla}\cdot(\vec{E}\times\vec{B}) + \vec{B}\cdot\underbrace{\vec{\nabla}\times\vec{E}}_{-\frac{\partial \vec{B}}{\partial t}}$$

by Maxwell's eqs.

Thus,

$$\int_V d^3x \; \vec{J}\cdot\vec{E} = - \int_V d^3x \left\{ \frac{1}{\mu_0} \vec{\nabla}\cdot(\vec{E}\times\vec{B}) + \right.$$

$$\left. + \frac{1}{\mu_0} \vec{B} \frac{\partial \vec{B}}{\partial t} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\cdot\vec{E} \right\}$$

Now remember that the energy density of EM field is given by

$$w = \frac{1}{2}\frac{1}{\mu_0}\vec{B}^2 + \frac{1}{2}\epsilon_0 \vec{E}^2$$

Thus, one may write

$$\int_V d^3x \; \vec{J}\cdot\vec{E} = \int_V \left\{ -\frac{\partial w}{\partial t} - \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B}) \right\}$$

Since the volume is arbitrary, one may rewrite this equation in a differential form,

$$-\vec{J}\cdot\vec{E} = \frac{\partial w}{\partial t} + \vec{\nabla}\cdot\vec{S}$$

where $$\boxed{\vec{S} = \frac{1}{\mu_0}\,\vec{E}\times\vec{B}}$$

is called the Poynting vector. It has the dimension (energy) / (area × time) and describes the energy flux.

For a plane wave:

$$\vec{E} = \mathrm{Re}\left\{ E_1 \vec{\epsilon}_1 + E_2 e^{i\varphi} \vec{\epsilon}_2 \quad e^{-i\omega t + i k \bar{x}} \right\} =$$

$$= \mathrm{Re}\left\{\left(E_1 \vec{\epsilon}_1 + E_2 (\cos\varphi + i \sin\varphi) \vec{\epsilon}_2 \right) (\cos\theta + i \sin\theta)\right\} =$$

$$\theta = -\omega t + k\bar{x}$$

$$= E_1 \cos\theta \, \vec{\epsilon}_1 + E_2 \cos(\theta + \varphi) \, \vec{\epsilon}_2$$

$$\vec{B} = \frac{k}{\omega} \vec{n} \times \vec{E}$$

$$\vec{E} \times \vec{B} = \frac{k}{\omega} \vec{E} \times (\hat{n} \times \vec{E}) =$$

$$= \frac{k}{\omega} \varepsilon_{ijk} E_j \quad \varepsilon_{kpq} n_p E_q =$$

$$= \frac{k}{\omega} \left(\delta_{ip} \delta_{jq} - \cancel{\delta_{iq} \delta_{jp}} \right) E_j n_p E_q =$$

$$= \frac{k}{\omega} \vec{n} \cdot E_j^2 = \frac{\vec{k}}{\omega} \cdot (\vec{E} \cdot \vec{E})$$

$$\vec{S} = \frac{\vec{k}}{\omega} \frac{1}{\mu_0} \left(E_1^2 \cos^2\theta + E_2^2 \cos^2(\theta + \varphi) \right)$$

averaging

$$\rightarrow \langle \vec{S} \rangle = \frac{1}{2} \frac{k}{\omega} \frac{1}{\mu_0} \left(E_1^2 + E_2^2 \right) = \frac{1}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} \left(E_1^2 + E_2^2 \right) \vec{n}$$

For the plane wave we have

$$\vec{S} = \frac{k}{\omega}\frac{1}{\mu_0}\vec{E}^2 = \sqrt{\frac{\epsilon_0}{\mu_0}}\;\vec{E}^2\cdot\vec{n}$$

while the energy density is

$$w = \frac{1}{2}\frac{1}{\mu_0}\vec{B}^2 + \frac{\epsilon_0}{2}\vec{E}^2 =$$

$$= \frac{1}{2}\frac{k^2}{\omega^2}\frac{1}{\mu_0}\vec{E}^2 + \frac{\epsilon_0}{2}\vec{E}^2 = \epsilon_0\vec{E}^2$$

Thus, $\vec{S} = \vec{n}\;\underbrace{c\cdot w}$

(velocity of light) × (energy density)

as it should be.

EM waves in the medium

In the absence of sources/currents, the Maxwell eqs in a medium read:

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \frac{1}{\mu}\vec{\nabla}\times\vec{B} - \epsilon\frac{\partial\vec{E}}{\partial t} = 0 \qquad (*)$$

$$\Downarrow$$

$$\vec{\nabla}\times\vec{B} - \mu\epsilon\frac{\partial\vec{E}}{\partial t} = 0$$

ϵ = electric permittivity

μ = magnetic permeability

These equations only differ from the Maxwell's eqs in the vacuum by the replacement $\mu_0 \to \mu$, $\epsilon_0 \to \epsilon$. Now recall that the product $\epsilon_0\mu_0$ determines the speed of light in the vacuum, as it enters the d'Alambert eqs for $\vec{E}$ and $\vec{B}$. In the medium we have

$$\left[\mu\epsilon\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right]\vec{E} = 0$$

$$\left[\mu\epsilon\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right]\vec{B} = 0$$

Thus, the speed of light is replaced by

$$c \to v = \frac{1}{\sqrt{\mu \epsilon}}$$

The ratio

$$n = \frac{c}{v} = \sqrt{\frac{\mu \epsilon}{\mu_0 \epsilon_0}}$$

is called the <u>refraction index</u>.

The general solution of the plane-wave type now becomes

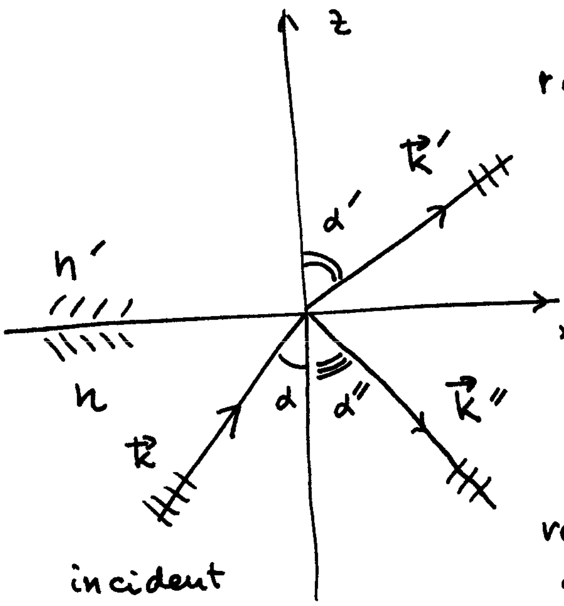
$$f_1(vt - x) + f_2(vt + x)$$

$\Rightarrow$ v is indeed the velocity of wave propagation

Since the behavior of EM waves is completely determined by the Maxwell's eqs, the properties that we have derived above remain valid, with the replacement $c \to v = \frac{1}{\sqrt{\mu \epsilon}}$.

Reflection and refraction

Properties of a medium (in particular, the refraction index) may change in space. The simplest case is an interface between two transparent media with different n. Consider the case of a flat interface



refracted wave:

$$\vec{E}' = \vec{E}_0' e^{-i\omega t + i\vec{k}'\vec{x}}$$

$$\vec{B}' = \sqrt{\mu'\epsilon'} \frac{\vec{k}' \times \vec{E}'}{|\vec{k}'|}$$

incident wave:

$$\vec{E} = \vec{E}_0 e^{-i\omega t + i\vec{k}\vec{x}}$$

$$\vec{B} = \sqrt{\mu\epsilon} \frac{\vec{k} \times \vec{E}}{|\vec{k}|}$$

reflected wave:

$$\vec{E}'' = \vec{E}_0'' e^{-i\omega t + i\vec{k}''\vec{x}}$$

$$\vec{B}'' = \sqrt{\mu\epsilon} \frac{\vec{k}'' \times \vec{E}''}{|k|}$$

The wave numbers satisfy the relations

$$|\vec{k}| = |\vec{k}''| \equiv k = \omega\sqrt{\mu\epsilon}$$

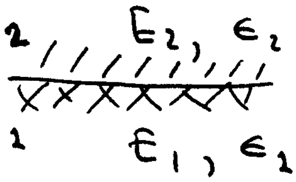
$$|\vec{k}'| \equiv k' = \omega\sqrt{\mu'\epsilon'}$$

Now let us derive relations between the characteristics of the incident, reflected and refracted waves. These relations follow from the necessity to satisfy boundary conditions at the interface between two media. We will assume that these media are dielectrics. The boundary conditions read (in the absence of external charges):

$$\vec{n}_{21} \times \left(\vec{E}_2 - \vec{E}_1 \right) = 0$$

$$\vec{n}_{21} \cdot \left(\varepsilon_2 \vec{E}_2 - \varepsilon_1 \vec{E}_1 \right) = 0$$

$$(*)$$



$$\vec{n}_{21} \left(\vec{B}_2 - \vec{B}_1 \right) = 0$$

$$\vec{n}_{21} \times \left(\frac{1}{\mu_2} \vec{B}_2 - \frac{1}{\mu_1} \vec{B}_1 \right) = 0$$

The boundary conditions, regardless of their particular form, have to be satisfied at all moments of time and in all points of the boundary. This is only possible if we requre that at $z=0$ the phase factors are equal,

$$(\vec{k}\cdot\vec{x})\Big|_{z=0} = (\vec{k}'\cdot\vec{x})\Big|_{z=0} = (\vec{k}''\cdot\vec{x})\Big|_{z=0} \qquad (*)$$

$\Rightarrow$ these eqs. imply that projections of $\vec{k}, \vec{k}', \vec{k}''$ on the plane (xy) are all parallel

$\Downarrow$

three vectors $\vec{k}$, $\vec{k}'$ and $\vec{k}''$ lie in the same plane orthogonal to (xy).

Eq. $(*)$ is then equivalent to

$$k \sin\alpha = k' \sin\alpha' = k'' \sin\alpha''$$

1) $k = k'' \quad \Rightarrow \quad \boxed{\alpha = \alpha''}$

2) $$\frac{\sin\alpha}{\sin\alpha'} = \frac{k'}{k} = \frac{\sqrt{\mu'\epsilon'}}{\sqrt{\mu\epsilon}} = \frac{n'}{n} \qquad \text{(Snell's law)}$$

Now assume that eqs (225*) are satisfied and consider the consequences of the boundary conditions (224*). In application to our problem they read:

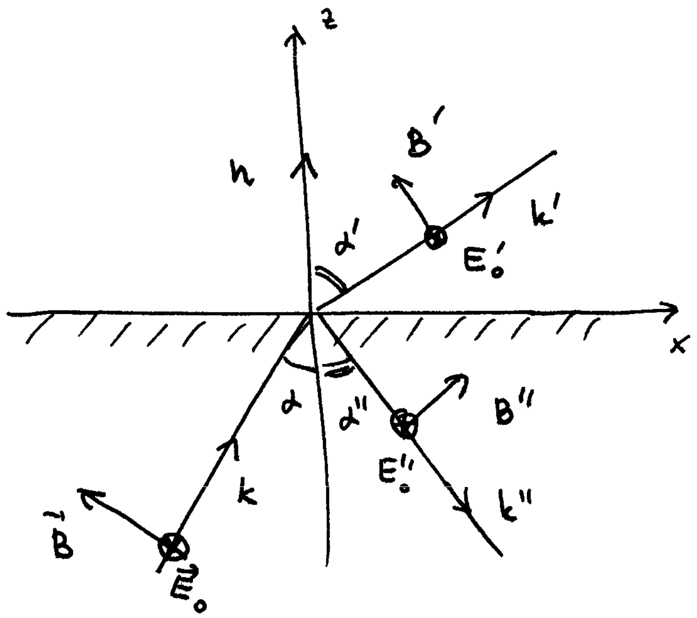
$$\boxed{\begin{aligned} &\vec{n} \times \left(\vec{E}_0' - \vec{E}_0 - \vec{E}_0'' \right) = 0 \\ &\vec{n} \cdot \left(\epsilon \left(\vec{E}_0 + \vec{E}_0'' \right) - \epsilon' \vec{E}_0' \right) = 0 \end{aligned}}$$

$$\vec{n} \cdot \left(\sqrt{\mu'\epsilon'} \cdot \frac{\vec{k}' \times \vec{E}_0'}{k'} - \sqrt{\mu\epsilon} \left(\frac{\vec{k} \times \vec{E}_0}{k} + \frac{\vec{k}'' \times \vec{E}_0''}{k''} \right) \right) = 0$$

$$\Rightarrow \quad \boxed{\vec{n} \cdot \left(\vec{k}' \times \vec{E}_0' - \vec{k} \times \vec{E}_0 - \vec{k}'' \times \vec{E}_0'' \right) = 0}$$

$$\boxed{\vec{n} \times \left(\frac{1}{\mu} \left(\vec{k} \times \vec{E}_0 + \vec{k}'' \times \vec{E}_0'' \right) - \frac{1}{\mu'} \left(\vec{k}' \times \vec{E}_0' \right) \right) = 0}$$

Consider first the case when the electric field is linearly polarised and is perpendicular to the plane of incidence.



Boundary conditions give:

1). $E_0' - E_0 - E_0'' = 0$

2). $0 = 0$

3) $k' E_0' \sin\alpha' - k E_0 \sin\alpha - k'' E_0'' \sin\alpha'' = 0$

(equivalent to 1))

4) $$\frac{1}{\mu}\left(k E_0 \cos\alpha - k'' E_0'' \cos\alpha\right) -$$

$$- \frac{1}{\mu'} k' E_0' \cos\alpha' = 0$$

now recall that

$$k = k'' = \sqrt{\mu\epsilon}\cdot\omega = n\frac{\omega}{c}$$
$$k' = \sqrt{\mu'\epsilon'}\cdot\omega = n'\frac{\omega}{c}$$
$$n = \frac{\sqrt{\mu\epsilon}}{\sqrt{\mu_0\epsilon_0}}$$

thus 4) $\rightarrow$

$$\frac{n}{\mu}\left(E_0 - E_0''\right)\cos\alpha - \frac{n'}{\mu'}E_0'\cos\alpha' = 0$$

since these eqs are linear in E, we cannot determine the absolute magnitude. $\Rightarrow$ Divide everything by E_0.

$$x_1 \equiv \frac{E_0'}{E_0}$$

1). $\rightarrow$ $\quad x_1 - x_2 = 1$

$$x_2 \equiv \frac{E_0''}{E_0}$$

4) $\rightarrow$ $\quad \frac{n}{\mu}\left(1 - x_2\right)\cos\alpha - \frac{n'}{\mu'}x_1\cos\alpha' = 0.$

$$x_1 = 1 + x_2$$

$$\frac{n}{\mu}\left(1-x_2\right)\cos\alpha = \frac{n'}{\mu'}\left(1+x_2\right)\cos\alpha'$$

$$\frac{n}{\mu}\cos\alpha - \frac{n'}{\mu'}\cos\alpha' = \left(\frac{n}{\mu}\cos\alpha + \frac{n'}{\mu'}\cos\alpha'\right)x_2$$

$$X_2 = \frac{n\cos\alpha - \frac{\mu}{\mu'} n'\cos\alpha'}{n\cos\alpha + \frac{\mu}{\mu'} n'\cos\alpha'}$$

$\cos\alpha' = ?$

$$\frac{\sin^2\alpha}{\sin^2\alpha'} = \frac{n'^2}{n^2}$$

$$\sin^2\alpha' = \frac{n^2}{n'^2}\sin^2\alpha$$

$$\cos\alpha' = \sqrt{1 - \frac{n^2}{n'^2}\sin^2\alpha} =$$

$$= \frac{1}{n'}\sqrt{n'^2 - n^2\sin^2\alpha}$$

$$\Rightarrow \boxed{\frac{E_0''}{E_0} = \frac{n\cos\alpha - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\alpha}}{n\cos\alpha + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\alpha}}}$$

$$\boxed{\frac{E_0'}{E_0} = 1 + X_2 = \frac{2n\cos\alpha}{n\cos\alpha + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\alpha}}}$$

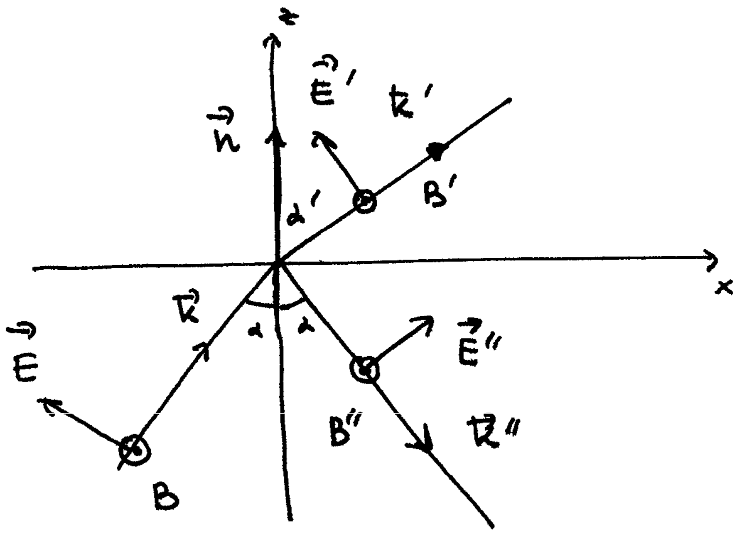
In optics one usually deals with the case $\frac{\mu}{\mu'} \approx 1$

Then

$$\frac{E_0'}{E_0} = \frac{2\cos\alpha}{\cos\alpha + \sqrt{n'^2/n^2 - \sin^2\alpha}}$$

$$\frac{E_0''}{E_0} = \frac{\cos\alpha - \sqrt{n'^2/n^2 - \sin^2\alpha}}{\cos\alpha + \sqrt{n'^2/n^2 - \sin^2\alpha}} \qquad (*)$$

Consider now the case of the electric field parallel to the plane of incidence.



In this case boundary conditions give:

1). $$E_0 \cos\alpha - E_0'' \cos\alpha - E_0' \cos\alpha' = 0$$

2). $$\epsilon E_0 \sin\alpha + \epsilon E_0'' \sin\alpha - \epsilon' E' \sin\alpha' = 0$$

3). $$0 = 0$$

4). $$\frac{1}{\mu} k E_0 + \frac{1}{\mu} k'' E_0'' - \frac{1}{\mu'} k' E_0' = 0$$

$$4) \rightarrow \quad \frac{n}{\mu} E_0 + \frac{n}{\mu} E_0'' - \frac{n'}{\mu'} E_0' = 0$$

$$2) : \quad \frac{\sqrt{\epsilon\mu}}{\mu} \cancel{\sqrt{\epsilon\mu}\,\sin\alpha} \left(E_0 + E_0''\right) - \frac{\sqrt{\epsilon'\mu'}}{\mu'} \cancel{\sqrt{\epsilon'\mu'} \times \sin\alpha'}\, E'_x = 0$$

$$\sqrt{\epsilon\mu}\,\sin\alpha = \sqrt{\epsilon'\mu'}\,\sin\alpha'$$

$$\frac{n}{\mu}\left(E_0 + E_0''\right) - \frac{n'}{\mu'} E' = 0 \quad \Rightarrow \text{equivalent to 4).}$$

Thus, we have two eqs:

$$a) \quad \cos\alpha\left(E_0 - E_0''\right) - \cos\alpha' \, E_0' = 0$$

$$b) \quad \frac{n}{\mu}\left(E_0 + E_0''\right) - \frac{n'}{\mu'} E_0' = 0$$

denote as before $\quad x_1 = \frac{E_0'}{E_0} \; ; \; x_2 = \frac{E_0''}{E_0}$

$$b) \rightarrow \quad \frac{n}{\mu}(1 + x_2) = \frac{n'}{\mu'} x_1$$

$$x_1 = \frac{n\mu'}{n'\mu}(1 + x_2)$$

$$a) \rightarrow \quad \cos\alpha\,(1 - x_2) = \frac{n\mu'}{n'\mu} \cos\alpha' \cdot (1 + x_2)$$

$$\frac{n'}{\mu'} \cos\alpha\,(1 - x_2) = \frac{n}{\mu} \cos\alpha'\,(1 + x_2)$$

$$\frac{n'}{\mu'}\cos\alpha - \frac{n}{\mu}\cos\alpha' = \left(\frac{n'}{\mu'}\cos\alpha + \frac{n}{\mu}\cos\alpha'\right) X_2$$

$$X_2 = \frac{\frac{n'}{\mu'}\cos\alpha - \frac{n}{\mu}\cos\alpha'}{\frac{n'}{\mu'}\cos\alpha + \frac{n}{\mu}\cos\alpha'}$$

Now recall that $\cos\alpha' = \frac{1}{n'}\sqrt{n'^2 - n^2\sin^2\alpha}$

$$\Rightarrow \quad X_2 = \frac{\frac{n'}{\mu'}\cos\alpha - \frac{n^2}{\mu n'}\sqrt{n'^2/n^2 - \sin^2\alpha}}{n'/\mu'\cos\alpha + \frac{n^2}{\mu n'}\sqrt{n'^2/n^2 - \sin^2\alpha}} =$$

$$= \frac{\frac{n'^2}{n^2}\cos\alpha - \frac{\mu'}{\mu}\sqrt{n'^2/n^2 - \sin^2\alpha}}{\frac{n'^2}{n^2}\cos\alpha + \frac{\mu'}{\mu}\sqrt{n'^2/n^2 - \sin^2\alpha}}$$

$$X_1 = \frac{n\mu'}{n'\mu}(1 + X_2) =$$

$$= \frac{n\mu'}{n'\mu} \cdot \frac{2\frac{n'^2}{n^2}\cos\alpha}{\frac{n'^2}{n^2}\cos\alpha + \frac{\mu'}{\mu}\sqrt{\frac{n'^2}{n^2} - \sin^2\alpha}}$$

$$= \frac{2\frac{n'}{n}\frac{\mu'}{\mu}\cos\alpha}{\frac{n'^2}{n^2}\cos\alpha + \frac{\mu'}{\mu}\sqrt{\frac{n'^2}{n^2} - \sin^2\alpha}}$$

If we neglect again $\mu'/\mu \simeq 1$, then finally

$$\boxed{\begin{aligned} \frac{E_0''}{E} &= \frac{\frac{n'^2}{n^2}\cos\alpha - \sqrt{\frac{n'^2}{n^2} - \sin^2\alpha}}{\frac{n'^2}{n^2}\cos\alpha + \sqrt{\frac{n'^2}{n^2} - \sin^2\alpha}} \\ \frac{E_0'}{E} &= \frac{2\,\frac{n'}{n}\cos\alpha}{\frac{n'^2}{n^2}\cos\alpha + \sqrt{\frac{n'^2}{n^2} - \sin^2\alpha}} \end{aligned}} \qquad (*)$$

Let us discuss the resulting expressions (229*) and (*).

① We see from eqs (*) that at a certain angle the amplitude of the reflected wave polarized parallel to the plane of incidence <u>vanishes</u>.

This happens at an angle when (taking $\mu'/\mu = 1$)

$$\frac{n'^2}{n^2}\cos\alpha = \sqrt{\frac{n'^2}{n^2} - \sin^2\alpha}$$

$$\frac{n'^4}{n^4}\cos^2\alpha = \frac{n'^2}{n^2} - \sin^2\alpha$$

denote: $\xi \equiv \dfrac{n'^2}{n^2}$

then

$$\xi^2 \cos^2\alpha = \xi - \sin^2\alpha$$

$$\xi^2 - \xi^2 \sin^2\alpha = \xi - \sin^2\alpha$$

$$\xi^2 - \xi = (\xi^2 - 1) \sin^2\alpha$$

$$\sin^2\alpha = \frac{\xi^2 - \xi}{\xi^2 - 1} = \frac{\xi}{\xi + 1}$$

$$\xi^2 \cos^2\alpha = \xi - 1 + \cos^2\alpha$$

$$\cos^2\alpha = \frac{\xi - 1}{\xi^2 - 1} = \frac{1}{\xi + 1}$$

$$\mathrm{tg}^2\alpha = \xi = \frac{n'^2}{n^2}$$

$$\mathrm{tg}\,\alpha = \frac{n'}{n} \Rightarrow \alpha = \mathrm{arctg}\left(\frac{n'}{n}\right)$$

The angle α is called Brewster's angle.

If incident light falls at the Brewster's angle, the reflected light is completely polarized with the polarization vector κ perpendicular to the plane of incidence.

② <u>Total internal reflection</u> happens for the case $n > n'$, i.e. when light propagates in from more dense to less dense medium.

The Snell's law

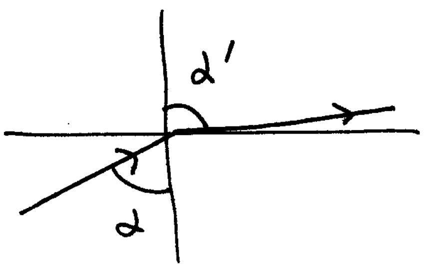
$$n \sin\alpha = n' \sin\alpha'$$

admits solutions for α' only as long as

$$\sin\alpha < \frac{n'}{n}$$

$$\alpha < \arcsin\left(\frac{n'}{n}\right)$$

When α approaches this value, α' approaches $\frac{\pi}{2}$.



Beyond that angle there is no refracted wave, only the reflected one.

One may show that for $\alpha > \arcsin\left(\frac{n'}{n}\right)$ the field in the second medium is exponentially dumped.

3. When the incidence angle α goes to 0, both polarizations go to the same values of refracted and reflected waves,

$$\frac{E_0'}{E_0} \rightarrow \frac{2n}{n'+n}$$

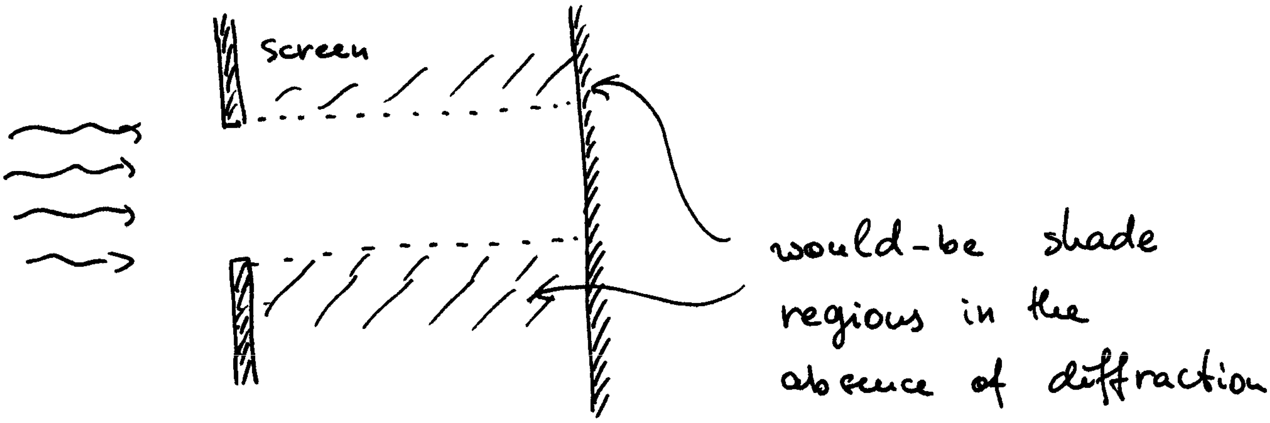
$$\frac{E_0''}{E_0} \rightarrow \frac{n'-n}{n'+n}$$

$\Rightarrow$ the reflection is proportional to the difference in the refraction indices of the two media.

Interference of waves. Diffraction

Everybody knows that light propagates along straight lines, (in a homogeneous medium). This is only true, however, in the limit of infinitely small wave length. The deviations from the laws of geometrical optics which are due to wave nature of light (or other electromagnetic radiation) are called diffraction phenomena.

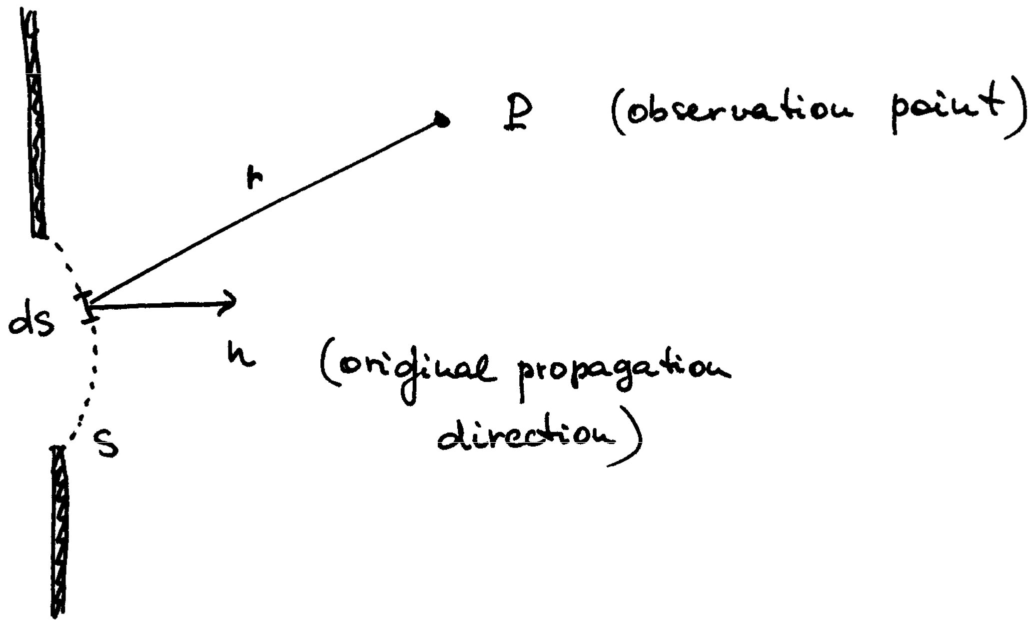
Typical diffraction problem:



The exact field configuration may be obtained by solving the field equations with appropriate boundary conditions imposed on the surface of the screen. This is often prohibitively complicated.

In many cases of practical interest, one may obtain approximate solutions by the method that is described below. This method is applicable in cases when deviations from geometrical optics are small: the sizes of objects (screens etc.) are large as compared to the wavelength and angles describing the deviations of light propagation from geometrical optics are small.

Consider a screen with a hole

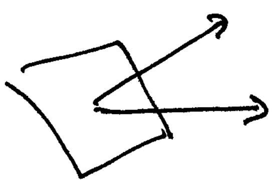


Let $f(\vec{x})$ denote any component of the electromagnetic fields $\vec{E}$ or $\vec{B}$ (without overall time-dependent phase $e^{-i\omega t}$).

Our aim is to determine $f(x)$ in a given observation point P.

The main idea of the method is based on the following principle (Huygens): imagine a surface S covering the hole in the screen. The field at P can be considered as sum (superposition) of contributions of individual elements $d\vec{s}$ of the surface S. (This statement is only approximately correct; for justification see Jackson).

More concretely, the field f at the point P is proportional to $d\vec{s}$ projected on the plane, perpendicular to the light propagation.



$$f_P \propto d\vec{s}\cdot\vec{n} \cdot f$$

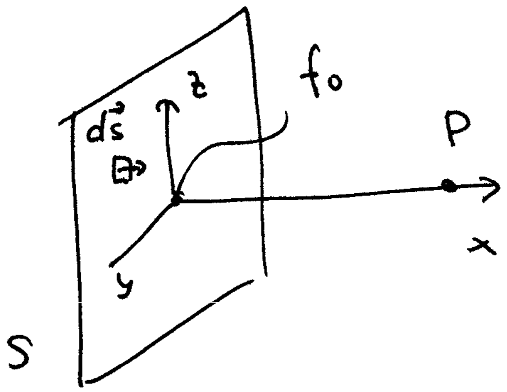
It is also proportional to the value of the field at the point of the surface element $d\vec{s}$.

Finally, one has to take into account the distance r between $d\vec{s}$ and P. Thus,

$$f_P = \text{const} \cdot \int \frac{e^{+ikr}}{r} \cdot f \cdot d\vec{s}\cdot\vec{n} \qquad (*)$$

[Note that together with the time-dependent phase, $\frac{1}{r} e^{-i\omega t + ikr}$ is a solution to the wave equation.]

Now we have to fix the normalization. To this end we require that eq (239*) reproduces the field of the plane wave in the absence of screens. Take the wave propagating in the direction of x.



$$f_P = f_0 \cdot e^{ikx} \qquad (*)$$

(because we have taken plane wave).

Compare (*) to the result of application of eq.(239*). S we take to be the plane (yz).

Then

$$d\vec{s}\cdot\vec{n} = dx\,dy$$

$$f = f_0$$

$$r = \sqrt{x^2+y^2+z^2} \simeq \sqrt{x^2\left(1+\frac{y^2+z^2}{x^2}\right)}$$

$$= x\left(1+\frac{1}{2}\frac{y^2+z^2}{x^2}+\dots\right)$$

$$= x + \frac{y^2+z^2}{2x} + \dots$$

$$f_P = \text{const} \int dz\,dy \, \frac{e^{ikx} e^{i\frac{z^2+y^2}{2x}}}{x+\dots} \cdot f_0$$

$$= \frac{\text{const}}{x} f_0 e^{ikx} \underbrace{\int dz\, e^{ik\frac{z^2}{2x}}} \int dy\, e^{ik\frac{y^2}{2x}} \overset{*}{=}$$

$$y = \sqrt{\frac{2x}{k}}\,\tilde{y}$$

$$z = \sqrt{\frac{2x}{k}}\,\tilde{z}$$

$$\int dz\, e^{ik\frac{z^2}{2x}} = \sqrt{\frac{2x}{k}} \underbrace{\int d\tilde{z}\, e^{i\tilde{z}^2}}_{I}$$

$$\overset{*}{=} \frac{\text{const}}{\cancel{x}} f_0 e^{ikx} \cdot \frac{2\cancel{x}}{k} I^2$$

Now calculate I^2:

$$I^2 = \int d\tilde{z}\, e^{i\tilde{z}^2} \cdot \int d\tilde{y}\, e^{i\tilde{y}^2} =$$

$$= \int d\tilde{y}\, d\tilde{z}\; e^{i(\tilde{z}^2 + \tilde{y}^2)} =$$

$$= 2\pi \int_0^\infty \rho\, d\rho\; e^{i\rho^2} =$$

$$= \pi \int_0^\infty d\xi\; e^{i\xi - \varepsilon\xi}$$

regulator to be sent to 0 at the end of calculation

$$= \pi \frac{1}{i-\varepsilon} e^{(i-\varepsilon)\xi} \Big|_0^\infty =$$

$$= \pi \frac{-1}{i-\varepsilon} \xrightarrow[\varepsilon\to 0]{} i\pi$$

Thus, we obtain

$$f_P = \text{const}\, \frac{2\pi i}{k} f_0 e^{ikx}$$

comparing to $f_P = f_0 e^{ikx}$ we find

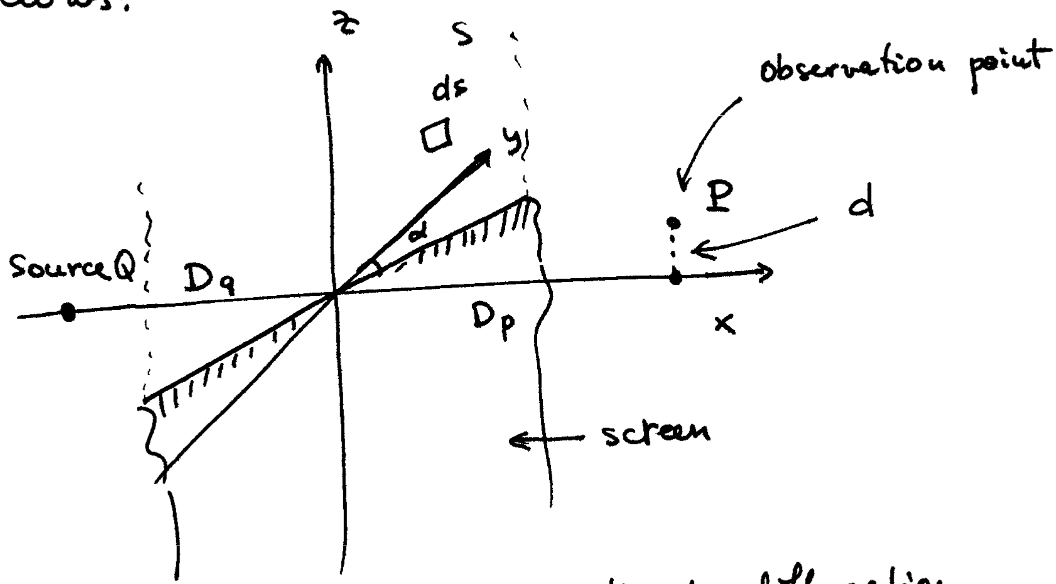
$$\text{const} = \frac{k}{2\pi i}$$

Thus, from (239*)

$$\boxed{f_P = \int \frac{k \cdot f_0}{2\pi i\, r} e^{ikr}\, d\vec{s}\cdot\vec{n}}$$

[For derivation from first principles see Jackson].

As an application of this relation, calculate the distribution of light intensity near the edge of the shade. When screens are large, each edge may be considered approximately as straight. Let us choose the coordinate system as follows:



without diffraction:

shade : $z < 0$

light : $z > 0$.

Screen edge is in the plane (xy) forming angle α with axis y.

D_q – source - screen distance (see fig.)

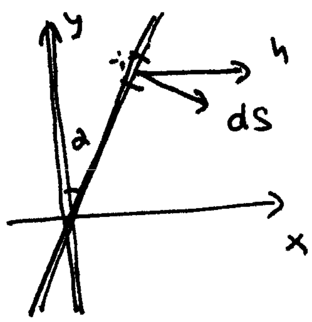
D_p – screen - observation point distance. (see fig).

S – continuation of the screen plane.

Consider contribution of a given element of the surface S with coordinates x, y, z.
The coordinates x and y are related by

$$x = y \cdot tg\alpha$$

view from the top:



$$d\vec{s} \cdot \vec{n} = dz \cdot \frac{dy}{\cos\alpha} \cdot \cos\alpha =$$

$$= dz\, dy$$

The incident field f at $d\vec{s}$ is proportional to

$$f \sim \frac{1}{\sqrt{y^2 + z^2 + (D_q + y\,tg\alpha)^2}}\; e^{ik \underbrace{\sqrt{y^2 + z^2 + (D_q + y\,tg\alpha)^2}}_{R_q}}$$

$$r = \sqrt{y^2 + (z - d)^2 + (D_p - y\,tg\alpha)^2}$$

Now we have to calculate

$$f_p = \int \frac{k}{2\pi i} \frac{1}{R_q \cdot r}\, e^{ik(R_q + r)}\, dy\,dz$$

To calculate this integral let us make some approximations. The main contribution to the integral comes from the region close to the origin. Therefore, we may take

$$y, z \ll D_p, D_q$$

In addition, the slowly-varying factors in front of the exponential may be approximately considered as constants. (but not the ones in the exponential!) Thus,

$$f_p \propto \int e^{ik(R_q + r)} \cdot dy\, dz$$

The distances R_q and r may be expanded in powers of y, z:

$$R_q = \sqrt{y^2 + z^2 + (D_q + y\,tg\alpha)^2} =$$

$$= (D_q + y\,tg\alpha)\sqrt{1 + \frac{y^2 + z^2}{(D_q + y\,tg\alpha)^2}} =$$

$$= (D_q + y\,tg\alpha)\left(1 + \frac{y^2 + z^2}{2(D_q + y\,tg\alpha)^2} + \ldots\right)$$

$$= D_q\left(1 + \frac{y\,tg\alpha}{D_q}\right)\left(1 + \frac{y^2 + z^2}{2D_q^2} + \ldots\right)$$

$$= D_q + y \operatorname{tg}\alpha + \frac{y^2 + z^2}{2(D_q + y \operatorname{tg}\alpha)} + \dots$$

$$= D_q + y \operatorname{tg}\alpha + \frac{y^2 + z^2}{2 D_q} + \dots$$

Likewise,

$$r = \sqrt{y^2 + (z-d)^2 + (D_p - y \operatorname{tg}\alpha)^2} =$$

$$= (D_p - y \operatorname{tg}\alpha) \sqrt{1 + \frac{y^2 + (z-d)^2}{(D_p - y \operatorname{tg}\alpha)^2}} =$$

$$= (D_p - y \operatorname{tg}\alpha) \left(1 + \frac{y^2 + (z-d)^2}{2(D_p - y \operatorname{tg}\alpha)^2} + \dots \right)$$

$$= D_p - y \operatorname{tg}\alpha + \frac{y^2 + (z-d)^2}{2 D_p}$$

Therefore

$$R_q + r = D_p + D_q + \frac{y^2 + z^2}{2 D_q} + \frac{y^2 + (z-d)^2}{2 D_p}$$

$$= \left[z\left(\frac{1}{D_p}+\frac{1}{D_q}\right)-\frac{d}{D_p}\right]^2 \frac{1}{1/D_p+1/D_q} \; -$$

$$\underbrace{-\frac{d^2}{D_p^2}\,\frac{1}{1/D_p+1/D_q}+\frac{d^2}{D_p}}\;=$$

$$+\;d^2\left[\frac{-D_q}{D_p\,(D_p+D_q)}+\frac{1}{D_p}\right]=d^2\,\frac{\cancel{D_p}}{\cancel{D_p}\,(D_p+D_q)}$$

$$= \frac{1}{1/D_p+1/D_q}\left[z\left(\frac{1}{D_p}+\frac{1}{D_q}\right)-\frac{d}{D_p}\right]^2+\frac{d^2}{D_p+D_q}$$

(irrelevant factor with magnitude = 1 (pure phase))

$$f_p = \text{const}\cdot e^{\frac{ikd^2}{2(D_p+D_q)}} \times$$

$$\times \int_0^\infty dz\,\exp\left\{\frac{ik\,D_pD_q}{2(D_p+D_q)}\left[z\left(\frac{1}{D_p}+\frac{1}{D_q}\right)-\frac{d}{D_p}\right]^2\right\}$$

$$f_p \;\alpha\; e^{ik(D_p+D_q)} \cdot \int dy\, e^{iky^2\left(\frac{1}{2D_p}+\frac{1}{2D_q}\right)}$$

$$\times \int dz\, e^{ik\left(\frac{z^2}{2D_q}+\frac{(z-d)^2}{2D_p}\right)}$$

The first factor is an overall phase which cancels in the intensity which is proportional to $|f_p|^2$. The second factor is d-independent; it gives a constant as far as d-dependence is concerned. Thus,

$$f_p = \text{const.} \int dz\, e^{ik\underbrace{\left(\frac{z^2}{2D_q}+\frac{(z-d)^2}{2D_p}\right)}}$$

$$\frac{z^2}{D_q} + \frac{z^2}{D_p} - 2\,\frac{zd}{D_p} + \frac{d^2}{D_p} =$$

$$= z^2\left(\frac{1}{D_q}+\frac{1}{D_p}\right) - 2\left(\frac{1}{D_q}+\frac{1}{D_p}\right)\frac{d/D_p}{1/D_q+1/D_p}\cdot z$$

$$+ \frac{d^2}{D_p} =$$

Let us now change variables

$$y^2 = \frac{k D_q D_p}{2(D_p+D_q)} \left[z\left(\frac{1}{D_p}+\frac{1}{D_q}\right) - \frac{d}{D_p}\right]^2$$

$$dz \rightarrow (\text{constant factor}) \times dy$$

when $z = 0$

$$y = -\frac{d}{D_p} \cdot \left(\frac{k D_q D_p}{2(D_p+D_q)}\right)^{1/2} =$$

$$= - d\sqrt{\frac{k D_q}{2D_p(D_p+D_q)}} \equiv -W$$

Assembling all factors we have:

$$\boxed{\begin{aligned} f_p &= \text{const} \cdot \int_{-w}^{\infty} dy\, e^{iy^2} \\ \text{where} \quad w &= d\sqrt{\frac{k D_q}{2D_p(D_p+D_q)}} \end{aligned}} \qquad (*)$$

The intensity of light in the point P is proportional to $|f_p|^2$,

$$I \propto \text{const} \cdot \left[\left(\int_{-w}^{\infty} \cos y^2\, dy\right)^2 + \left(\int_{-w}^{\infty} \sin y^2\, dy\right)^2\right]$$

The integrals which appear here are similar to those that we have already calculated, except for the limits. We found

$$\int_{-\infty}^{\infty} \cos y^2 \, dy = \sqrt{\frac{\pi}{2}} = \int_{-\infty}^{\infty} \sin y^2 \, dy$$

One introduces the so-called Fresnel integrals,

$$C(z) = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{z}} \cos y^2 \, dy$$

$$S(z) = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{z}} \sin y^2 \, dy$$

They often arise in diffraction problems.

Obviously, at $z \to \infty$ one has

$$C(z) \to 1/2$$

$$S(z) \to 1/2$$

Making use of this property one may finally write

$$\boxed{I = \frac{1}{2} I_0 \left\{ \left(C(w^2) + 1/2 \right)^2 + \left(S(w^2) + 1/2 \right)^2 \right\}} \quad (*)$$

At $w \to \infty$ ($d \to \infty$) we have $I = I_0$, the intensity without any screen.

Let us discuss the physical meaning of eq. (250*).

In the region of the shade,

$$w < 0 .$$

Consider a limiting case of large $|w|$.

$$\int_{|w|}^{\infty} e^{i\eta^2} d\eta = -\frac{1}{2i|w|} e^{iw^2} + \frac{1}{2i} \int_{|w|}^{\infty} e^{i\eta^2} \frac{d\eta}{\eta^2}$$

$$= e^{iw^2} \left\{ -\frac{1}{2i|w|} + \frac{1}{4|w|^3} - \dots \right\}$$

Taking the first term and neglecting the rest, we obtain

$$I = \frac{I_0}{4\pi w^2}$$

$\Rightarrow$ In the region of would-be shade the intensity rapidly falls off, as the distance to the shade boundary squared.

The characteristic distances are determined by

$$w \sim 1$$

$$\Rightarrow \quad d \sqrt{\frac{k D_q}{2 D_p (D_q + D_p)}} \sim 1$$

Let for definiteness $D_q \gg D_p$. Then this condition becomes

$$\sqrt{\frac{k\, d^2}{2\, D_p}} \sim 1$$

$$k d \cdot \frac{d}{D_p} \sim 1$$

Since $k^{-1} \sim \lambda$, this gives

$$d \sim \sqrt{D_p \cdot \lambda}$$

Consider now the illuminated region. It corresponds to $w > 0$. Then we can write

$$\int_{-w}^{\infty} e^{i\eta^2}\, d\eta = \int_{-\infty}^{\infty} e^{i\eta^2}\, d\eta - \int_{-\infty}^{-w} e^{i\eta^2}\, d\eta =$$

$$= (1+i)\sqrt{\frac{\pi}{2}} - \int_{w}^{\infty} e^{i\eta^2}\, d\eta$$

$$\xrightarrow[w \gg 1]{} (1+i)\sqrt{\frac{\pi}{2}} + \frac{1}{2iw}\, e^{iw^2}$$

Making use of eq. (249*) one may write

$$I = \frac{I_0}{2} \left| \sqrt{\frac{2}{\pi}} \int_{-w}^{\infty} e^{i\eta^2} d\eta \right|^2 =$$

$$= \frac{1}{2} I_0 \left| 1 + i + \frac{1}{\sqrt{2\pi}\, i w} (\cos w^2 + i \sin w^2) \right|^2$$

$$= \frac{1}{2} I_0 \left| 1 + \frac{1}{\sqrt{2\pi} \cdot w} \sin w^2 + i \left(1 - \frac{1}{\sqrt{2\pi}\, w} \cos w^2 \right) \right|^2$$

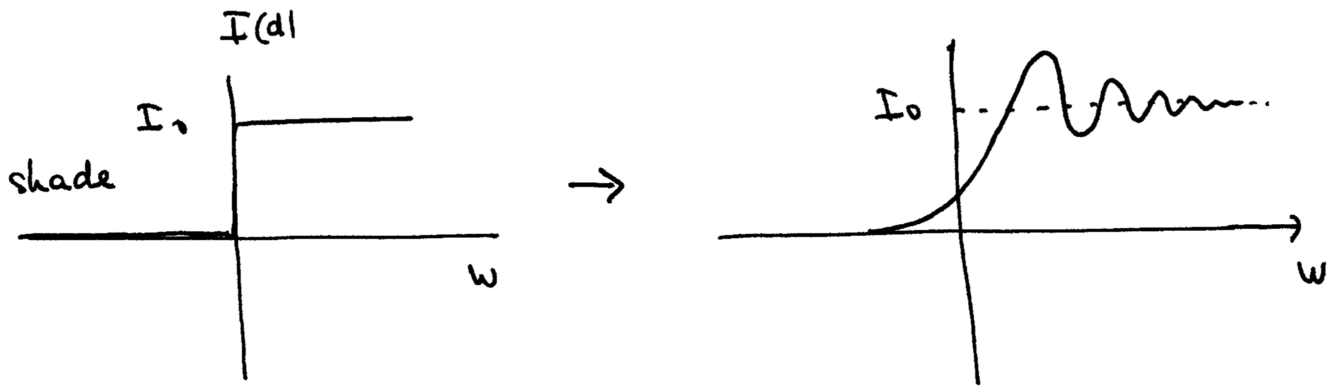
$$|a+ib|^2 = a^2 + b^2$$

$$= \frac{1}{2} I_0 \left\{ \left(1 + \frac{1}{\sqrt{2\pi}\, w} \sin w^2 \right)^2 + \left(1 - \frac{1}{\sqrt{2\pi}\, w} \cos w^2 \right)^2 \right\} =$$

$$= I_0 \left\{ 1 + \frac{1}{\sqrt{2\pi}\, w} \left(\sin w^2 - \cos w^2 \right) + \ldots \right\}$$

$$= I_0 \left\{ 1 + \frac{1}{\sqrt{\pi}} \frac{1}{w} \sin \left(w^2 - \frac{\pi}{4} \right) \right\}$$

This is an oscillating function with the decreasing amplitude.



Emission of EM waves

I. Field of a moving source. So far we have considered either electric and magnetic fields produced by static charges and stationary currents, or time-dependent EM fields in the vacuum (EM waves). Consider now the fields of moving soures (charges & currents).

Maxwell's eqs:

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho$$

$$\vec{\nabla}\times\vec{B} - \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J}$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \qquad (*)$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t}$$

defs. of EM potentials. (**)

The last two of eqs. (255*) are automatically satisfied if one plugs in the definitions of potentials:

$$③ \to \quad \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

$$④: \quad \vec{\nabla} \times \left(- \cancel{\vec{\nabla}\varphi} - \frac{\partial \vec{A}}{\partial t} \right) + \frac{\partial}{\partial t} \vec{\nabla} \times \vec{A} =$$

$$= \quad - \frac{\partial}{\partial t} \vec{\nabla} \times \vec{A} + \frac{\partial}{\partial t} \vec{\nabla} \times \vec{A} = 0. \qquad \text{ok}$$

Consider now the inhomogeneous equations, the vector equation first:

$$\vec{\nabla} \times (\overbrace{\vec{\nabla} \times \vec{A}}^{\vec{B}}) - \frac{1}{c^2} \frac{\partial}{\partial t} \overbrace{\left(- \vec{\nabla}\varphi - \frac{\partial}{\partial t} \vec{A} \right)}^{\vec{E}} = \mu_0 \vec{J}$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = - \Delta \vec{A} + \vec{\nabla} (\vec{\nabla} \vec{A}) \qquad \text{(check it!)}$$

Thus, we have:

$$\left[- \Delta + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \vec{A} + \vec{\nabla} \left\{ \vec{\nabla} \vec{A} + \frac{1}{c^2} \frac{\partial \varphi}{\partial t} \right\} =$$

$$= \mu_0 \vec{J}$$

Now recall that the potentials are not defined uniquely: to the vector potential A_μ one may add a 4-gradient of a scalar function,

$$A_\mu \to A_\mu + \partial_\mu \lambda$$

We may use this gauge freedom to impose one condition on 4 functions φ, $\vec{A}$. Let us choose the Lorentz condition,

$$\partial_\mu A^\mu = 0 \qquad \partial_\mu = \left(\tfrac{1}{c}\partial_t , \partial_i \right) \qquad A^\mu = \left(\varphi , cA_i \right)$$

$$\downarrow$$

$$\boxed{\vec{\nabla}\vec{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0} \qquad (v)$$

With this condition imposed on the potentials, the vector equation becomes simply

$$\boxed{\left\{ -\Delta + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right\} \vec{A} = \mu_0 \vec{J}} \qquad (*)$$

Consider now the first of eqs. (255*):

$$\vec{\nabla}\underbrace{\left(-\vec{\nabla}\varphi - \frac{\partial}{\partial t}\vec{A}\right)} = \frac{1}{\epsilon_0}\rho$$

$$-\vec{\nabla}^2\varphi - \underbrace{\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}} =$$

$$= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi$$ by the Lorentz condition (257 v)

Thus

$$\boxed{-\Delta\varphi + \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = \frac{1}{\epsilon_0}\rho} \qquad (*)$$

This eq. has the same form as (257*), the form of inhomogeneous d'Alambert equation. It is valid for time-dependent ρ as well as for static one. We need now to solve this equation. A general solution is a <u>sum</u> of a general solution of the homogeneous eqs. (EM waves) + a particular solution of the inhomogeneous eq.

Let us solve eq. (258*). We make use of the linearity of this eq. and first consider a small volume dV occupied by the total charge

$$de = \rho \, dV$$

where $de = de(t)$ may change with time. Let us find the contribution of this elementary charge to the total potential. The equation becomes (replace our charge by the point charge):

$$-\Delta \varphi + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varphi = \frac{de(t)}{\epsilon_0} \cdot \delta^3(\vec{x}) \qquad (*)$$

The source on the r.h.s. is spherically symmetric, so we may look for spherically symmetric solutions. Let $r = |\vec{x}|$. Then

$$\Delta = \frac{1}{r^2} \partial_r r^2 \partial_r + (\text{angular part})$$

If $\varphi = \varphi(r)$, angular part $= 0$, and $(*)$ gives

$$-\frac{1}{r^2} \partial_r r^2 \partial_r \varphi + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varphi = \frac{de(t)}{\epsilon_0} \delta^3(x)$$

at $r \neq 0$ $\delta^3(x) = 0$, so the equation becomes

$$-\frac{1}{r^2}\partial_r r^2 \partial_r \varphi + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi = 0$$

To solve this eq. change variables

$$\varphi = \frac{\chi}{r}$$

$$\partial_r \varphi = \frac{\chi'}{r} - \frac{\chi}{r^2}$$

$$r^2 \partial_r \varphi = r\chi' - \chi$$

$$\partial_r r^2 \partial_r \varphi = r\chi'' + \chi' - \chi' = r\chi''$$

$$-\frac{1}{r}\chi'' + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{\chi}{r} = 0$$

$$-\chi'' + \frac{\partial^2}{c^2\partial t^2}\chi = 0$$

$$\chi = f_1(ct - r) + f_2(ct + r)$$

Recall that we are looking for a particular solution. Thus,

$$\varphi = \frac{1}{r}\chi(ct - r)$$

This is not all, because now we have to satisfy eq. (259*) at the origin $r=0$. The potential φ is singular at this point; it should be understood as a distribution (i.e., its integral with a smooth function should be finite). The second term in eq. (259*) is then not important as

$$\varphi \propto \frac{1}{r}$$

$$\int_{|x|\leq\varepsilon} d^3x \;\; \varphi \propto 4\pi \int_0^{\varepsilon} r^2 dr \; \frac{1}{r} \propto r^2 \Big|_0^{\varepsilon} \to 0$$

as $\varepsilon \to 0$

Without the second term

$$\text{eq (259*)} \quad \rightarrow \quad -\Delta\varphi = \frac{de(t)}{\epsilon_0}\,\delta^3(x)$$

This is a familiar Poisson equation.

$$-\Delta \frac{\chi(ct-r)}{r} \simeq -\chi(ct)\,\Delta\frac{1}{r} = \frac{de(t)}{\epsilon_0}\,\delta^3(x).$$

$$-\Delta\frac{1}{r} = 4\pi\,\delta^3(\bar{x})$$

$$\chi(ct) = \frac{1}{4\pi\epsilon_0}\, de(t) \qquad \text{(Columb law!)}$$

Finally, the solution is

$$\varphi(x,t) = \frac{1}{4\pi\epsilon_0} \cdot \frac{de(t - r/c)}{r} \qquad (*)$$

$\Rightarrow$ the scalar potential produced by the charge $de(t)$ is determined by the Columb law, but the charge should be taken in an <u>earlier</u> moment of time. The potential $(*)$ is called <u>retarded potential</u>.

We can now write a particular solution of eq. (258 $*$) in the case of arbitrary time-dependent charge density $\rho(\bar{x},t)$:

$$\boxed{\varphi(\bar{x},t) = \frac{1}{4\pi\epsilon_0}\int d\bar{x}'\, \frac{1}{|\bar{x}-\bar{x}'|}\cdot \rho\left(\vec{x}', t - \frac{|\bar{x}-\bar{x}'|}{c}\right)}$$

Since equation for the vector potential, eq. (257*), is essentially identical to (258*), we can write the solution immediately,

$$\boxed{\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \frac{1}{|\bar{x}-\bar{x}'|}\vec{J}\left(\bar{x}', t-\frac{|\bar{x}-\bar{x}'|}{c}\right)}$$

<u>Field of a single moving charge</u>. Let us calculate the field of a single charge e moving along the trajectory $\vec{r}_0(t)$. We have

$$\rho(\vec{x},t) = e\,\delta^3(\vec{x}-\vec{r}_0(t))$$

and thus

$$\varphi(\vec{x},t) = \frac{e}{4\pi\epsilon_0}\,\times$$

$$\times\int\frac{d^3x'}{|\bar{x}-\bar{x}'|}\,\delta^3\left(\vec{x}'-\vec{r}_0\left(t-\frac{|\vec{x}-\vec{x}'|}{c}\right)\right)$$

The integration may be simplified by introducing an extra δ-function to get rid of the complicated argument of $\delta^3(\ldots)$:

$$\varphi(\bar{x},t) = \frac{e}{4\pi\epsilon_0} \int \frac{d^3x'}{|\bar{x}-\bar{x}'|} \times$$

$$\times \int d\tau \; \delta\left(\tau - t + \frac{|\bar{x}-\bar{x}'|}{c}\right) \delta^3\left(\bar{x}' - \vec{r}_0(\tau)\right)$$

Now we can calculate $\int d^3x'$:

$$\varphi(\bar{x},t) = \int d\tau \; \frac{\delta\left(\tau - t + \frac{|\vec{x}-\vec{r}_0(\tau)|}{c}\right)}{|\bar{x} - \vec{r}_0(\tau)|} \cdot \frac{e}{4\pi\epsilon_0} \qquad (*)$$

Denote $\vec{R} = \bar{x} - \vec{r}_0(\tau) \qquad R = |\vec{R}|$

R = distance from a point on the trajectory to the observation point

The argument of the δ-function: the equation

$$t - \tau = \frac{|\vec{x} - \vec{r}_0(\tau)|}{c}$$

defines the time moment τ at which the light has to be emitted to reach point $\vec{x}$ precisely at the moment t:

$$c(t-\tau) = |\vec{x} - \vec{r}_0(\tau)|$$

(i.e. interval between $(ct, \vec{x})$ and $(c\tau, \vec{r}_0(\tau))$ is zero.)

The integral in (264*) equals:

$$\varphi(\bar{x}, t) = \frac{1}{4\pi\epsilon_0} \frac{1}{R} \frac{e}{\left| \frac{\partial}{\partial\tau}\left(\tau - t + |\vec{R}|/c\right) \right|} \overset{*}{=}$$

$$\frac{\partial R}{\partial \tau} = - \frac{\bar{v}\bar{R}}{R}$$

$$\overset{*}{=} \frac{1}{4\pi\epsilon_0} \frac{1}{R} \frac{e}{1 - \frac{\bar{v}\bar{R}}{cR}}$$

Finally,

$$\varphi(\bar{x},t) = \frac{e}{4\pi\epsilon_0} \frac{1}{R - \frac{\bar{v}\bar{R}}{c}} \qquad (*)$$

Here $\vec{v} = \frac{d\vec{r}_0}{dt}$ is particle velocity, and $\vec{R} = \bar{x} - \vec{r}_0(\tau)$, while the moment τ is determined from the equation

$$t - \tau = \frac{1}{c} |\vec{x} - \vec{r}_0(\tau)| . \qquad (**)$$

Analogously, for the vector potential we get

$$\vec{A}(\bar{x},t) = \frac{\mu_0 e}{4\pi} \cdot \frac{\vec{v}}{R - \frac{\vec{v}\vec{R}}{c}} \qquad (v)$$

The potentials $(*)$ and (v) are called the <u>Liénard - Viechert potentials</u>.

When calculating electric and magnetic fields from these potentials one has to keep in mind that r.h.s. of eqs. (266*) and (v) depend on the time τ related to observation time t by eq. (266 **). The derivative with respect to t is related to the derivative with respect to τ as follows:

$$\frac{\partial}{\partial t}\cdot(266**) \Rightarrow c\left(1-\frac{\partial\tau}{\partial t}\right)=\frac{\partial}{\partial t}\,|\overbrace{\vec{x}-\vec{r}_0(\tau)}^{\vec{R}}|=$$

$$=-\underbrace{\frac{\partial\vec{r}_0}{\partial\tau}}_{\vec{v}}\cdot\vec{R}\,\frac{1}{R}\frac{\partial\tau}{\partial t}=-\frac{\vec{v}\vec{R}}{R}\cdot\frac{\partial\tau}{\partial t}$$

$$\Rightarrow \boxed{\frac{\partial\tau}{\partial t}=\frac{1}{1-\frac{\vec{v}\cdot\vec{R}}{cR}}}$$

$$\frac{\partial}{\partial x^i}\,(266**) \Rightarrow$$

$$-c\,\frac{\partial\tau}{\partial x^i}=\overbrace{\frac{R_i}{R}-\frac{\bar{v}\bar{R}}{R}\frac{\partial\tau}{\partial x^i}}^{\partial_i R}$$

$$\Rightarrow \boxed{\nabla_i\tau=\frac{\partial\tau}{\partial x^i}=-\frac{R_i}{cR-\bar{v}\bar{R}}}$$

Denoting $\Delta = cR - \bar{v}\bar{R}$ we write these relations as

$$\frac{\partial \tau}{\partial t} = \frac{cR}{\Delta}$$

$$\partial_i \tau = - \frac{R_i}{\Delta}$$

<u>Calculation of $\vec{E}$</u> :

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}$$

$$\varphi = \frac{e}{4\pi c \epsilon_0} \cdot \frac{1}{\Delta}$$

$$A_i = \frac{\mu_0 e}{4\pi c} \cdot \frac{v_i}{\Delta}$$

$$\partial_i R = \frac{R_j}{R} \partial_i R_j = \frac{R_j}{R} \partial_i \left(x_j - r_{oj}(\tau) \right) =$$

$$= \frac{R_j}{R} \left(\delta_{ij} + \dot{r}_{oj} \frac{R_i}{\Delta} \right) =$$

$$= \frac{R_i}{R} + \frac{\bar{v}\bar{R}}{R\Delta} R_i =$$

$$= \frac{R_i}{} \left(1 + \frac{\bar{v}\bar{R}}{} \right) = \frac{cR_i}{}$$

$$\partial_i \Delta = \partial_i \left(cR - \vec{v}\vec{R} \right) =$$

$$= c^2 \frac{R_i}{\Delta} + \dot{\vec{v}} \cdot \vec{R} \frac{R_i}{\Delta} - v_j \partial_i R_j =$$

$$= \left(c^2 + \dot{\vec{v}}\vec{R} \right) \frac{R_i}{\Delta} - v_j \left\{ \delta_{ij} + v_j \frac{R_i}{\Delta} \right\}$$

$$= \left(c^2 - v^2 + \dot{\vec{v}}\vec{R} \right) \frac{R_i}{\Delta} - v_i$$

①: $$-\nabla_i \varphi = + \frac{e}{4\pi c\, \epsilon_0} \frac{1}{\Delta^2} \partial_i \Delta =$$

$$= \frac{e}{4\pi c\, \epsilon_0} \frac{1}{\Delta^2} \left\{ \left(c^2 - v^2 + \dot{\vec{v}}\vec{R} \right) \frac{R_i}{\Delta} - v_i \right\}$$

$$\frac{\partial}{\partial t} R = \frac{\partial}{\partial t} \left| \bar{x} - \vec{r}_0(\tau) \right| = \frac{R_j}{R} \partial_t R_j =$$

$$= \frac{R_j}{R} \left(- v_j \frac{\partial \tau}{\partial t} \right) = - \frac{R_j}{R} v_j \frac{cR}{\Delta} =$$

$$= - \frac{c \cdot \bar{v}\bar{R}}{\Delta}$$

$$\partial_t R_j = - v_j \frac{cR}{\Delta}$$

$$\partial_t \Delta = \partial_t \left(cR - \bar{v}\bar{R} \right) =$$

$$= - c^2 \frac{\bar{v}\bar{R}}{\Delta} - \frac{\dot{v}_j}{} R_j \cdot \frac{cR}{\Delta} -$$

$$- v_j \left(- v_j \frac{cR}{\Delta} \right) =$$

$$= - c^2 \frac{\bar{v}\bar{R}}{\Delta} + v^2 \frac{cR}{\Delta} - \dot{\vec{v}}\vec{R} \cdot \frac{cR}{\Delta}$$

② : $$- \frac{\partial A_i}{\partial t} = - \frac{\mu_0 e}{4\pi c} \frac{\partial}{\partial t} \frac{v_i}{\Delta} =$$

$$= - \frac{\mu_0 e}{4\pi c} \left\{ \dot{v}_i \frac{cR}{\Delta^2} - \frac{v_i}{\Delta^2} \left[- c^2 \frac{\bar{v}\bar{R}}{\Delta} + \right.\right.$$

$$\left.\left. + v^2 \frac{cR}{\Delta} - \dot{\vec{v}}\bar{R} \cdot \frac{cR}{\Delta} \right] \right\}$$

$$E_i = - \nabla_i \phi - \frac{\partial}{\partial t} A_i =$$

$$= \frac{e}{4\pi c \epsilon_0} \frac{1}{\Delta^2} \left\{ \left(c^2 - v^2 + \dot{\vec{v}}\vec{R} \right) \frac{R_i}{\Delta} - v_i \right.$$

$$- \dot{v}_i \frac{1}{c} R - v_i \frac{\bar{v}\bar{R}}{\Delta} + v_i \frac{v^2}{c^2} \frac{cR}{\Delta} -$$

$$\left. - \frac{v_i \dot{\vec{v}}\vec{R}}{c^2} \frac{cR}{\Delta} \right\}$$

$$= \frac{e}{4\pi c\,\epsilon_0}\;\frac{1}{\Delta^2}\left\{\left(1-\frac{v^2}{c^2}\right)\frac{1}{\Delta}\left(c^2 R_i - v_i c R\right)\right.$$

$$\left. + \;\dot{v}\bar{R}\cdot\frac{R_i}{\Delta} - \frac{\dot{v}_i}{c}R - \frac{v_i R\,\dot{v}\bar{R}}{c\Delta}\right\} \tag{$*$}$$

The magnetic field can be shown to have the form

$$B = \frac{1}{cR}\,\vec{R}\times\vec{E}$$

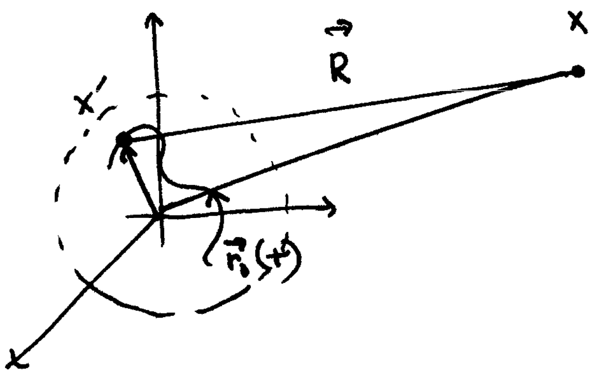
i.e. always perpendicular to $\vec{E}$

Expression (*) is complicated. One important property is clear, however. $\vec{E}$ consists of two parts. First part depends on the particle velocity and decays as $\frac{1}{r^2}$ at large distances. It is the field of a uniformly moving charge.

Second part is proportional to particle acceleration; it decays as $\frac{1}{r}$ far from the moving charge. These are electromagnetic waves emitted by the charge.

EM field at large distances

Expression (271*) is cumbersome, so we have to find an approximation. A useful approximation exists when the size of the emitting system is much smaller than the distance to the observation point.



In the expressions for the retarded potentials, the distance $|\bar{x} - \bar{x}'|$ can be approximated as

$$|\bar{x} - \bar{x}'| = \sqrt{\bar{x}^2 - 2\bar{x}\bar{x}' + \bar{x}'^2} \simeq$$

$$\simeq x\sqrt{1 - 2\frac{xx'}{x^2}} \simeq$$

$$\simeq x - \bar{n}\cdot\bar{x}'$$

where $\vec{n}$ is the unit vector in the direction of $\vec{x}$,

$$\vec{n} = \frac{\vec{x}}{|x|}$$

Then the retarded potentials become

$$\varphi(x,t) = \frac{1}{4\pi\epsilon_0} \int \frac{dx'}{x - \bar{n}\bar{x}'} \rho\left(\bar{x}', t - \frac{x}{c} + \frac{\bar{n}\bar{x}'}{c}\right)$$

$$\simeq \frac{1}{4\pi\epsilon_0} \frac{1}{x} \int dx' \rho\left(\bar{x}', t - \frac{x}{c} + \frac{\bar{n}\bar{x}'}{c}\right)$$

$$\vec{A}(x,t) \simeq \frac{\mu_0}{4\pi \cdot x} \int d^3x' \, \vec{J}\left(\bar{x}', t - \frac{x}{c} + \frac{\bar{n}\bar{x}'}{c}\right) \qquad (*)$$

Here we have neglected x' as compared to x in denominators. Whether the terms containing x' in the argument of ρ and $\vec{J}$ can be neglected as well depends on how rapidly these functions change in time $\Delta t \sim \frac{\bar{n}\bar{x}'}{c}$. This approximation holds if $v \ll c$ (charges move with non-relativistic velocities).

If the distance $|x|$ is large as compared to the wavelength of the emitted wave, one may approximately consider the wave as plane. Then the electric and magnetic fields are related as

$$\vec{E} = c \cdot \vec{B} \times \vec{n}$$

and it is sufficient to calculate

$$\vec{B} = \vec{\nabla} \times \vec{A} =$$

$$= \vec{\nabla}_x \times \left\{ \frac{\mu_0}{4\pi x} \int d^3x' \, \vec{J}\left(\bar{x}', t - \frac{x}{c} + \frac{\bar{n}\bar{x}'}{c}\right) \right\}$$

$$= \frac{\mu_0}{4\pi} \left\{ \underbrace{- \frac{\vec{n}}{x^2} \times \int d^3x' \, \vec{J}(\dots)}_{\propto \frac{1}{x^2}} \right. +$$

the only important term

$$+ \left. \frac{\mu_0}{4\pi x} \int d^3x' \, \dot{\vec{J}}(\bar{x}', t - \dots) \cdot \nabla_x \left(t - \frac{x}{c} + \underbrace{\frac{\vec{n}\vec{x}'}{c}}_{\to \frac{1}{x^2}}\right) \right\}$$

$$= - \frac{\mu_0}{4\pi x} \, \frac{\vec{n}}{c} \times \int d^3x' \, \dot{\bar{J}}(\bar{x}', t \dots)$$

$$= \frac{1}{c} \left[\dot{\vec{A}} \times \vec{n} \right].$$

Here we have neglected the terms $\frac{1}{x^2}$ as compared to $\frac{1}{x}$. The final expression is the same as in the plane wave.

Thus, at large distances from the source we have

$$\vec{B} = \frac{1}{c} \cdot \dot{\vec{A}} \times \vec{n}$$

$$\vec{E} = [\dot{\vec{A}} \times \vec{n}] \times \vec{n} \qquad (*)$$

with $\vec{A}$ determined by eq. (273*).

Dipole emission

As has been mentioned above, expressions (273*) can be simplified further if during the time interval $\Delta t = \frac{\bar{n} \cdot \bar{x}'}{c}$ the positions of charges do not substantially change (this happens when $\lambda \ll a$, the system size, or, equivalently, $v \ll c$).

Consider a system of charges that move in a finite region of space, and find the EM field at large distances from the charges. If charges are non-relativistic, one may neglect the time $\frac{\bar{n}\,\bar{x}'}{c}$ in eqs. (273*) which then become

$$\vec{A}(\bar{x},t) = \frac{\mu_0}{4\pi\, X} \int d^3x' \; \vec{J}\left(\bar{x}', t - \frac{X}{c}\right)$$

Since $\vec{J}\left(\bar{x}', t - \frac{X}{c}\right) = \vec{v}\cdot\rho\left(x', t - \frac{X}{c}\right)$

we get

$$\vec{A}(x,t) = \frac{\mu_0}{4\pi\, X} \cdot \underbrace{\sum e_i \vec{v}_i\left(t - \frac{X}{c}\right)}_{\frac{d}{dt}\sum_i e_i \vec{r}_i\left(t-\frac{X}{c}\right) = \dot{\vec{d}}\left(t - \frac{X}{c}\right)} \stackrel{*}{=}$$

$(\vec{d} = \text{dipole moment.})$

$$\stackrel{*}{=} \frac{\mu_0}{4\pi\, X}\, \dot{\vec{d}}$$

One can now use eqs. (275*) to obtain electric and magnetic fields in this approximation,

$$\vec{B} = \frac{\mu_0}{4\pi c \cdot X} \cdot \ddot{\vec{d}} \times \vec{n}$$

$$\vec{E} = \frac{\mu_0}{4\pi\ X} \cdot (\ddot{\vec{d}} \times \vec{n}) \times \vec{n}$$

Note that $\ddot{\vec{d}}$ depends on the <u>accelerations</u> of charges, not on velocities. The charges moving with constant velocity do not radiate.

The total energy carried away by the electromagnetic waves can be found by calculating the Poynting vector

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{\vec{n}}{16\pi^2 X^2} \frac{\mu_0}{c} \cdot |\ddot{\vec{d}} \times n|^2$$

$$= \frac{\vec{n}}{16\pi^2} \cdot \frac{\mu_0}{c} \cdot \frac{|\ddot{\vec{d}}|^2}{X^2} \sin^2\theta$$

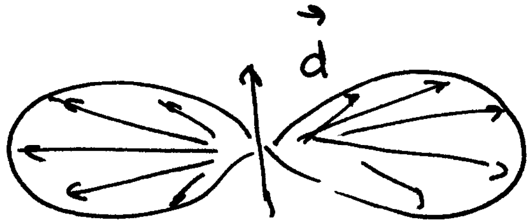
where θ is the angle between $\ddot{\vec{d}}$ and $\vec{n}$

Remarks:

- the intensity of radiation is proportional to the square of the second derivative of dipole moment, $|\ddot{\vec{d}}|^2$

- the intensity of radiation is inversely proportional to the square of the distance x^2.

- the radiation is directional (not isotropic). In the case when $\vec{d}/d$ remains constant,



- the total amount of radiation per unit time is

$$I = \frac{\mu_0 |\ddot{\vec{d}}|^2}{16\pi^2 c} \int \underbrace{\sin^2\theta \ \sin\theta \, d\theta}_{\frac{4}{3}} \cdot \underbrace{d\varphi}_{2\pi} = \frac{\mu_0}{6\pi c} |\ddot{\vec{d}}|^2$$

For a single charge with acceleration $\vec{a}$

$$I = \frac{\mu_0 e^2}{6\pi c} \cdot a^2$$